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NOTE ON THE LINEAR SYMMETRIC CONGRUENCE IN n VARIABLES

L. J. MORDELL

Introduction. Let $f = f(x_1, x_2, \ldots, x_n)$ be a polynomial in $n > 2$ variables with integer coefficients, and let p be a large prime. Very little appears to be known about estimates for the number N of solutions of the congruence

$$f \equiv 0 \pmod{p} \tag{1}$$

for general f. For the special case when

$$f = a_1 x_1^{l_1} + \ldots + a_n x_n^{l_n} + a, \tag{1'}$$

I showed [1, p. 207] in 1932, that when $a_1 a_2 \ldots a_n a \not\equiv 0 \pmod{p}$,

$$N = p^{n-1} + O(p^{\frac{1}{2}(n-1)})$$

where the constants implied in O are independent of the a's. Particular cases of (1′) have been discussed by many writers even as far back as Gauss, and (1′) has been the subject of numerous papers in recent years. But no other instances of (1) seem to have been considered and so it may be of interest to study the case when f is the general linear symmetric function of the x's, say

$$f = a_0 + a_1 \sum x_1 + a_2 \sum x_1 x_2 + \ldots + a_n x_1 x_2 \ldots x_n \equiv 0. \tag{2}$$

I have been led to the conjectured

THEOREM. *When $f \equiv 0$ is not derived from $x_1 x_2 \ldots x_n \equiv a$ by a linear substitution $x_r' \equiv (Ax_r + B)/(Cx_r + D)$ $(r = 1, 2, \ldots, n)$, nor from any of*

$$a\, x_1 x_2 \ldots x_n + b \sum x_1 x_2 \ldots x_{n-1} \equiv 0,$$

$$\sum x_1 x_2 \ldots x_{n-2} \equiv 0, \qquad \sum x_1 x_2 \ldots x_{n-3} \equiv 0, \qquad \ldots, \sum x_1 x_2 \equiv 0,$$

by a linear substitution $x_r' \equiv Ax_r + B$ $(r = 1, 2, \ldots, n)$, then N, the number of solutions of (2), *satisfies*

$$N = p^{n-1} + O(p^{\frac{1}{2}(n-1)}), \tag{3}$$

where the constant implied in O is independent of the a's.

The proofs for $n = 2, 3$ are trivial. For $n = 4$, the result lies deep since it is practically equivalent to Hasse's result [2, p. 145] that the number of solutions of the congruence

$$y^2 \equiv ax^4 + bx^3 + cx^2 + dx + e \tag{4}$$

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is $p + O(p^{\frac{1}{2}})$ provided that the quartic is not congruent to a multiple of the square of a quadratic.

I know of no really worth while results for $n > 4$.

Take the cases in turn.

$n = 2$:

(5) $$f = a_0 + a_1(x_1 + x_2) + a_2x_1x_2 \equiv 0.$$

It may be supposed that a_1 and a_2 are not both congruent to zero. Then obviously if $a_2 \equiv 0$, $N = p$. If $a_2 \not\equiv 0$, two cases arise according as f is decomposable or indecomposable. In the first case $a_1^2 - a_0a_2 \equiv 0$, and f becomes $(a_1 + a_2x_1)\cdot(a_1 + a_2x_2) \equiv 0$, and there are $2p - 1$ solutions. In the second $a_1^2 - a_0a_2 \not\equiv 0$, and there are $p - 1$ solutions since x_1 may have any of the $p - 1$ values for which $a_2x_1 + a_1 \not\equiv 0$, and then x_2 is uniquely determined.

$n = 3$:

(6) $$f = a_0 + a_1(x_1 + x_2) + a_2x_1x_2 + x_3(a_1 + a_2(x_1 + x_2) + a_3x_1x_2.$$

Let N' be the number of solutions in x_1, x_2 of

(7) $$a_1 + a_2(x_1 + x_2) + a_3x_1x_2 \equiv 0.$$

Suppose first that this is reducible. Then $a_3 \not\equiv 0$ and we can put $a_1 \equiv t^2a_3$, $a_2 \equiv ta_3$, and $N' = 2p - 1$. But then

$$f = a_0 - a_3t^3 + a_3(x_1 + t)(x_2 + t)(x_3 + t),$$

and this is one of the excluded cases. Clearly $N = (p - 1)^2$ if $a_0 - a_3t^3 \not\equiv 0$, but $N = 3p^2 - 3p + 1$ if $a_0 - a_3t^3 \equiv 0$.

Suppose then (7) is not reducible. Now $N' = p - 1$ or p, according as $a_3 \not\equiv 0$ or $a_3 \equiv 0$. For each of the remaining $p^2 - N'$ sets for x_1, x_2 there is a unique value for x_3. For the N' sets, there will be solutions for x_3, and then p solutions for x_3, if and only if

(8) $$\begin{aligned} a_1 + a_2(x_1 + x_2) + a_3x_1x_2 &\equiv 0, \\ a_0 + a_1(x_1 + x_2) + a_2x_1x_2 &\equiv 0. \end{aligned}$$

Suppose first that these congruences are essentially the same, say

$$\frac{a_1}{a_0} \equiv \frac{a_2}{a_1} \equiv \frac{a_3}{a_2} \equiv t.$$

Then $f \equiv a_0(1 + tx_1)(1 + tx_2)(1 + tx_3)$, and this is an excluded case. If we had had $a_0 \equiv a_1 \equiv a_2 \equiv 0$, then $f \equiv a_3x_1x_2x_3$, an excluded case.

Apart from these cases, (8) determine $x_1 + x_2$ and x_1x_2 uniquely and so there are $O(1)$ values for x_1 and x_2. Hence

$$N = p^2 - p + \epsilon + O(p) = p^2 + O(p), \qquad \epsilon = 0, 1.$$

The work shows that there is no exception to the general result (3) except in the case $a(x_1 + t)(x_2 + t)(x_3 + t) \equiv 0$.

$n = 4$:

Here $f = g + hx_4$, where

$$
(9)\qquad
\begin{aligned}
g &= a_0 + a_1(x_1 + x_2 + x_3) + a_2(x_2x_3 + x_3x_1 + x_1x_2) + a_3x_1x_2x_3,\\
h &= a_1 + a_2(x_1 + x_2 + x_3) + a_3(x_2x_3 + x_3x_1 + x_1x_2) + a_4x_1x_2x_3.
\end{aligned}
$$

The number N' of sets x_1, x_2, x_3 for which $h \equiv 0$ is $p^2 + O(p)$ except when $a_3 \equiv ta_4$, $a_2 \equiv t^2a_4$, $a_1 \equiv t^3a_4$, $a_4 \not\equiv 0$, and then f takes the excluded form

$$a_0 - a_4t^4 + a_4(x_1 + t)(x_2 + t)(x_3 + t)(x_4 + t).$$

Hence for $p^3 - p^2 - O(p)$ sets of values for x_1, x_2, x_3, there is a unique value for x_4. When $h \equiv 0$, we have solutions for x_4, in fact p solutions, if and only if x_1, x_2, x_3 satisfy both $g \equiv 0$, $h \equiv 0$, say for M solutions x_1, x_2, x_3. Then

$$N = p^3 - p^2 - O(p) + pM = p^3 + O(p^{3/2}),$$

as it will be shown that $M = p + O(p^{\frac{1}{2}})$.

Some of the exceptional cases arising are dealt with by the

LEMMA. *The functions g, h in* (9) *yield an identical congruence in x_1, x_2, x_3 of the form*

$$(10)\qquad h + g\lambda \equiv \mu(x_1 + k)(x_2 + k)(x_3 + k)$$

for constants λ, μ, k, only in some of the exceptional cases.

For (10) gives

$$
\begin{aligned}
a_4 + \lambda a_3 &\equiv \mu, & a_3 + \lambda a_2 &\equiv \mu k,\\
a_2 + \lambda a_1 &\equiv \mu k^2, & a_1 + \lambda a_0 &\equiv \mu k^3.
\end{aligned}
$$

Then

$$
\begin{aligned}
ka_4 - a_3 + \lambda(ka_3 - a_2) &\equiv 0,\\
ka_3 - a_2 + \lambda(ka_2 - a_1) &\equiv 0,\\
ka_2 - a_1 + \lambda(ka_1 - a_0) &\equiv 0.
\end{aligned}
$$

Put $f' = f(x_1, x_2, x_3, -k)$. Then

$$
\begin{aligned}
f' = g - kh &= (a_3 - ka_4)\, x_1x_2x_3 + (a_2 - ka_3)(x_2x_3 + x_3x_1 + x_1x_2) + \dots\\
&= (ka_1 - a_0)(\lambda x_1 - 1)(\lambda x_2 - 1)(\lambda x_3 - 1).
\end{aligned}
$$

Hence f can be written as

$$f = (x_4 + k)\, j + (ka_1 - a_0)(\lambda x_1 - 1)(\lambda x_2 - 1)(\lambda x_3 - 1),$$

where j is linear in each of x_1, x_2, x_3; and this expression for f must be symmetrical in x_1, x_2, x_3, x_4. Now $\lambda \equiv 0$ leads to an excluded form for f since if $k = 0$, $j = ax_1x_2x_3$. When $\lambda \not\equiv 0$, on replacing x_1 by $(x_1 + 1)/\lambda$ etc., f can be written as, say, $f = (x_4 + k')\, j' + ax_1x_2x_3$. Put

$$j' = a'x_1x_2x_3 + b'\sum_{1,2,3} x_2x_3 + c'\sum_{1,2,3} x_1 + d'.$$

Then

$$f = a'x_1x_2x_3x_4 + (a'k' + a)\,x_1x_2x_3 + b'(x_2x_3x_4 + x_3x_1x_4 + x_1x_2x_4) + c'(x_1x_4 + x_2x_4 + x_3x_4) + b'k'(x_2x_3 + x_3x_1 + x_1x_2) + c'k'(x_1 + x_2 + x_3) + d'x_4 + k'd'.$$

Since f is symmetrical in the x's, we have $b' = a'k' + a$, $c' = b'k'$, $d' = c'k'$, and so

$$f = a'\,x_1x_2x_3x_4 + b'\sum x_1x_2x_3 + b'k'\sum x_1x_2 + b'k'^2\sum x_1 + b'k'^3.$$

If $k' \equiv 0$, we have $f = a'\,x_1x_2x_3x_4 + b'\sum x_1x_2x_3$.

If $k' \not\equiv 0$, on multiplying f by k', we can clearly write f as

$$k'f = -\,a\,x_1x_2x_3x_4 + b'\,(x_1 + k')(x_2 + k')(x_3 + k')(x_4 + k').$$

Both these are exceptional cases. This concludes the proof of the lemma.

We now find the number M of solutions of $g \equiv 0$, $h \equiv 0$. Suppose first $a_4 \not\equiv 0$. Then if in f we replace x_1 by $x_1 - a_3/a_4$, etc., we may suppose $a_3 \equiv 0$, and so (9) takes for $h \equiv 0$ and $g \equiv 0$, the shape

$$\begin{aligned} x_1x_2x_3 &\equiv A\sum x_1 + B,\\ E\sum x_1x_2 &\equiv C\sum x_1 + D. \end{aligned} \tag{11}$$

If $E \equiv 0$, these can be written as

$$\sum x_1 \equiv F, \quad x_1x_2x_3 \equiv G,$$

where, by the lemma, $G \not\equiv 0$. Then

$$x_1x_2(F - x_1 - x_2) \equiv G.$$

This cannot be a reducible congruence; and writing $x_1 = X + Y$, $x_2 = X - Y$, the number of solutions is as in Hasse's result on (3). Hence we can suppose $E = 1$, and so, with $x_1 - c$ for x_1, etc., the two congruences (11) can be written as

$$\begin{aligned} x_1x_2x_3 &\equiv A\sum x_1 + B,\\ \sum x_1x_2 &\equiv D. \end{aligned} \tag{12}$$

Then

$$\begin{aligned} x_3(x_1x_2 - A) &\equiv A(x_1 + x_2) + B,\\ x_3(x_1 + x_2) &\equiv D - x_1x_2, \end{aligned}$$

and so

$$(A(x_1 + x_2) + B)(x_1 + x_2) + (x_1x_2 - A)(x_1x_2 - D) \equiv 0. \tag{13}$$

If (13) is reducible, $x_1 + x_2$ must be expressible linearly in terms of x_1x_2, and so $-As^2 - Bs + \frac{1}{4}(A - D)^2$ must be congruent to a perfect square in s, and so $B^2 + A(A - D)^2 \equiv 0$. Hence

$$A \equiv -A_1^2, \quad B \equiv A_1^3 + DA_1,$$

say. Then (12) becomes

$$x_1x_2x_3 \equiv -A_1{}^2\sum x_1 + A_1{}^3 + DA_1,$$
$$\sum x_1x_2 \equiv D.$$

By addition, $(x_1 - A_1)(x_2 - A_1)(x_3 - A_1) \equiv 0$, and this has been noted in the lemma and is dealt with in (14) below.

Write (13) as

$$(x_2{}^2 + A)\, x_1{}^2 + (2Ax_2 + B - (A + D)\, x_2)\, x_1 + Ax_2{}^2 + Bx_2 + AD \equiv 0.$$

Hence $2(x_2{}^2 + A)\, x_1 + 2Ax_2 + B - (A + D)\, x_2 = Y$, where $Y^2 \equiv R$ and R is a quartic in x_2. This quartic cannot be a perfect square since (13) is irreducible, and so the number of solutions is given by Hasse's result and (3) holds.

Suppose next $a_4 \equiv 0$. If $a_3 \not\equiv 0$, on replacing x_1 by $x_1 + k$, etc., we may suppose $a_2 = 0$ in f. Then (9) becomes

$$\text{(14)} \qquad \begin{aligned} g &= a_0 + a_1(x_1 + x_2 + x_3) + a_3\, x_1\, x_2\, x_3 \equiv 0, \\ h &= a_1 + a_3(x_2\, x_3 + x_3\, x_1 + x_1\, x_2) \equiv 0. \end{aligned}$$

These are the same as (12) and so we have the same estimate as when $a_4 \not\equiv 0$, except possibly when $g\lambda + h$ is reducible. This occurs only when $a_1 \equiv t^2 a_3$ and $a_0 \equiv 0$. Then

$$g + th \equiv a_3(x_1 + t)(x_2 + t)(x_3 + t),$$
$$g - th \equiv a_3(x_1 - t)(x_2 - t)(x_3 - t),$$

and so the solutions of (14) are given by $x_1 \equiv \pm t$, $x_2 \equiv \mp t$, x_3 arbitrary etc. Then $f = 0$ becomes

$$\sum x_1x_2x_3 + t^2 \sum x_1 \equiv 0.$$

When $t \equiv 0$, this is $\sum x_1\, x_2\, x_3 \equiv 0$. When $t \not\equiv 0$, on replacing x_1 by tx_1, etc., this becomes

$$\sum x_1 + \sum x_1x_2x_3 \equiv 0.$$

Suppose finally that $a_3 \equiv a_4 \equiv 0$ so that (9) becomes

$$\text{(15)} \qquad \begin{aligned} a_0 + a_1 \sum x_1 + a_2 \sum x_1x_2 &\equiv 0, \\ a_1 + a_2 \sum x_1 \qquad\qquad &\equiv 0. \end{aligned}$$

If $a_2 \equiv 0$, f becomes $a_0 + a_1(x_1 + x_2 + x_3 + x_4) \equiv 0$, and has p^3 solutions. If $a_2 \not\equiv 0$, on replacing x_1 by $x_1 + c$, etc., in (15), we have

$$x_1 + x_2 + x_3 + A \equiv 0, \quad x_2\, x_3 + x_3x_1 + x_1\, x_2 + B \equiv 0,$$

say, or

$$x_1\, x_2 + (x_1 + x_2)(-A - x_1 - x_2) + B \equiv 0.$$

Unless $A \equiv B \equiv 0$, this congruence is irreducible and has $p + O(1)$ solutions.

When $A \equiv B \equiv 0$, (15) is tantamount to $a_1 \equiv a_0 \equiv 0$, and $f = 0$ becomes

$$x_4(x_1 + x_2 + x_3) + x_2\, x_3 + x_3\, x_1 + x_1\, x_2 \equiv 0.$$

This is an excluded case which has $p^2(p-1)+p\,M'$ solutions, where M' is the number of solutions of

$$x_1+x_2+x_3\equiv 0,\quad x_2x_3+x_3x_1+x_1x_2\equiv 0$$

and so of

$$x_2^2+x_2x_3+x_3^2\equiv 0.$$

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ON THE WARING-SIEGEL THEOREM

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1. Introduction. The Waring problem deals with the decomposition of integers into sums of kth powers. Consider

$$\nu = \lambda_1^k + \ldots + \lambda_s^k. \tag{1}$$

Waring conjectured and Hilbert [**2**] first proved the existence of s depending on k only, such that every rational integer could be expressed as a sum of s kth powers.

It was Hardy and Littlewood [**1**] using the now classical "circle" method who obtained a bound for s as a function of k and at the same time derived an asymptotic formula for the number of solutions of (1). They proved the following theorem:

Let $C(\nu)$ be the number of solutions of (1) and let $s > (k-2)2^k + 5$, then

$$C(\nu) = \sigma_{\nu,k,s}\, \nu^{-1+s/k} \frac{\Gamma^s(1+1/k)}{\Gamma(s/k)} + o(\nu^{-1+s/k}),$$

where $\sigma_{\nu,k,s}$, the so-called singular series, is proved positive. It was then of interest to find the best possible result for the bound on s and at the same time to make the summands more general replacing in (1) kth powers by polynomial summands.

It was not until Hecke had developed the theory of theta functions in algebraic fields that Siegel [**8**; **9**; **10**; **11**] envisaged the possibility of extending the problem to algebraic fields. He proved a result (to be stated later) which corresponds to the above result of Hardy and Littlewood. It is our object to give two natural extensions of Siegel's theorem, namely to replace the kth powers by polynomial summands and to give a slight improvement of the lower bound for s. We rely for the most part on the methods of Siegel referring frequently as well to the methods of Landau [**7**] and Hua [**3**; **4**, **5**; **6**].

2. Notations, definitions, and formulation of the problem. Let F be an algebraic extension of the rationals of degree n and suppose that F is totally real, i.e. that all the conjugates of F are real. Let J be the ring of integers of F and suppose that $\omega_1, \ldots, \omega_n$ form a basis for J. If $\mathfrak{d}$ be the different (ramification ideal) for F and $(\omega^{(\wp)})^{-1} = (\rho^{(\wp)})$, then $\rho_1, \ldots, \rho_n$ is a basis for $\mathfrak{d}^{-1}$. The fundamental property of $\mathfrak{d}^{-1}$ used here is that if α is in $\mathfrak{d}^{-1}$, then $S(\lambda\alpha)$ is a rational integer for every λ in J. $S(\alpha)$ and $N(\alpha)$ denote as usual the trace and norm of α

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respectively. An inequality between elements of F means that it holds for all the corresponding conjugates, e.g. $\alpha < \beta$ means

$$\alpha^{(i)} < \beta^{(i)} \qquad i = 1, \ldots, n.$$

The Waring problem in algebraic fields has a somewhat different character from that in the rational field as shown by the following simple example. Let $R(\sqrt{d})$ be a quadratic field with $d \equiv 2, 3 \pmod 4$. The integers of such a field are of the form $a + b\sqrt{d}$, with a and b rational integers. The square of such an integer has even second coefficient; therefore an integer with odd second coefficient is never a sum of squares. This leads Siegel to the following construction; J_k is the ring generated by kth powers of elements of J. Finally let D be the discriminant of F. Consider now equation (1) as an equation in F with ν and λ_i totally positive. Let $B(\nu)$ be the number of solutions of (1) in F. Siegel [**10**] proved the following theorem:

If $s \geqslant kn(2^{k-1} + n) + 1$ then

$$B(\nu) = D^{\frac{1}{2}(1-s)} \sigma_{\nu,k,s} \left(\frac{\Gamma^s(1 + 1/k)}{\Gamma(s/k)}\right)^n N(\nu)^{-1+s/k} + o(N(\nu)^{-1+s/k}),$$

where $\sigma_{\nu,k,s} > 0$ if ν belongs to J_k and $\sigma_{\nu,k,s} = 0$ otherwise.

Consider now the polynomial

$$\phi(\xi) = \alpha\xi^k + \alpha_1\xi^{k-1} + \ldots + \alpha_k \tag{2}$$

with $\alpha, \alpha_i > 0$, $\nu > 0$ and

$$\nu = \phi(\xi_1) + \ldots + \phi(\xi_s). \tag{3}$$

It is our object to prove the following

Main Theorem. *Let* $A(\nu)$ *be the number of solutions of* (3) *and* $s \geqslant n(2^k + n) + 1$, *then*

$$A(\nu) = D^{\frac{1}{2}(1-s)} \sigma'_{\nu,k,s} \left(\frac{\Gamma^s(1 + 1/k)}{\Gamma(s/k)}\right)^n N(\alpha)^{-s/k} N(\nu)^{-1+s/k} + o(N(\nu)^{-1+s/k}).$$

We shall defer the discussion of the singular series to a further paper. It will be noted that the bound on s is a slight improvement over the previously known one but is far from the desirable bound which is independent of the degree of the field. For $k = 2$ Siegel has shown that such indeed is the case.

3. The generalized Farey dissection. Let X denote n-dimensional Euclidean space, then α in F is represented in X by the point $(\alpha^{(1)}, \ldots, \alpha^{(n)})$. If $(x_1, \ldots, x_n)$ be a point of X we put

$$\xi^{(j)} = \rho_1^{(j)} x_1 + \ldots + \rho_n^{(j)} x_n. \tag{4}$$

For γ in F, denote by $\mathfrak{a}_\gamma$ (dropping the subscript when it is clear what is meant) the denominator of $\gamma\mathfrak{d}$. Let $k > 1$ (no restriction) and $a = (2^{k-1} + n)$; let T satisfy $T^{2a} > 2D^{1/n}$, and put $t = T^{1-a}$, $h = T^{k-a-1}$. The O and o processes refer

to $T \to \infty$. For γ in F define B_γ as follows: B_γ is the set of points of X satisfying

$$N(\max(h|\xi - \gamma|, t^{-1})) < N(\mathfrak{d})^{-1}. \tag{5}$$

The B_γ are generalizations of the so-called "major arcs" and evidently B_γ is empty if $N(\mathfrak{a}) > t^n$.

THEOREM 3.1. *If $\gamma \neq \delta$, then $B_\gamma \cap B_\delta$ is empty.*

Proof. Suppose $B_\gamma \cap B_\delta \neq 0$; put

$$\max(h|\xi - \gamma|, t^{-1}) = \sigma^{-1}, \qquad \max(h|\xi - \delta|, t^{-1}) = \tau^{-1};$$

then $\sigma < t$, $\tau < t$, $(N\mathfrak{d}_\gamma \mathfrak{d}_\delta) < N(\sigma\tau)$. Moreover

$$|\gamma - \delta| < |\xi - \gamma| + |\xi - \delta| < h^{-1}(\sigma^{-1} + \tau^{-1}) < h^{-1}(\sigma + \tau)\,\sigma^{-1}\,\tau^{-1} < 2t(h\sigma\tau)^{-1};$$

therefore $N((\gamma - \delta)\,\mathfrak{a}_\gamma\,\mathfrak{a}_\delta) < 2^n t^n h^{-n} = 2^n T^{n-ns-kn-na+n} < D^{-1} < 1$. This is a contradiction since $(\gamma - \delta)\,\mathfrak{a}_\gamma\,\mathfrak{a}_\delta$ is an integral ideal.

THEOREM 3.2. *Let x be a point of X not in any B_γ, then there exist an integer α in F and a number β in $\mathfrak{d}^{-1}$ such that*

(i) $|\alpha\xi - \beta| < h^{-1}$, $\qquad 0 < |\alpha| < h$,

(ii) $\max(h|\alpha\xi - \beta|, |\alpha|) \geq D^{-\frac{1}{2}}$,

(iii) $\max(|\alpha^{(1)}|, \dots, |\alpha^{(n)}|) > t$,

(iv) $N(\alpha\beta\mathfrak{d})) < D^{\frac{1}{2}}$.

Proof. The proof may be found in Siegel [**10**]. This is the analogue of the usual theorem for the "minor arc" but is much more complicated. The proof is achieved by a multifold application of Minkowski's theorem on linear forms.

4. Analytical expression of $A(\nu)$. Let γ run over all numbers of F and let E denote the unit cube $0 \leq x_i < 1$ $(i = 1, \dots, n)$. Let E_0 be the set of points of E which do not lie in any B_γ.

Choose now a complete system of modulo $\mathfrak{d}^{-1}$ incongruent numbers γ with $N(\mathfrak{a}_\gamma) \leq t^n$. Denote this set by Γ; henceforth the summation index γ will range over the set Γ. If G be any group of transformations of a space into itself, we say that two points x, y of the space are equivalent with respect to G if there is a transformation of G taking x into y. A subset M of the space is called a fundamental region if no two points of M are equivalent and if every point of the space is equivalent to a point of M. Two subsets are equivalent if every point of the one is equivalent to a point of the other and conversely. The set of translations $\xi \to \xi + \rho$, with ρ any number of $\mathfrak{d}^{-1}$ forms a group H; E is clearly a fundamental region with respect to H.

THEOREM 4.1. *The sum of all B_γ summed over the set Γ is under H equivalent to $E - E_0$.*

Proof. Let ξ be a point of $E - E_0$, then there exists a number β in $\mathfrak{d}^{-1}$ and a number γ in Γ such that $\xi - \beta = \eta$ lies in B_γ. The disjointness of the B_γ provides then the uniqueness of γ and β.

Let $\mathscr{T}$ be the set of integers λ of F satisfying

$$0 < \lambda < T. \tag{6}$$

Let

$$f(x) = \sum_{\lambda \in \mathscr{T}} e(S(\phi(\lambda)\xi) \tag{7}$$

where $e(x)$ is an abbreviation for $e^{2\pi i x}$. Consider the following integral:

$$I = \int_E f^s(x)\, e(-S(\nu\xi))\, dx = \int_E g(x)\, dx,$$

say. On writing $f^s(x)$ as a multiple sum and using the properties of $\mathfrak{d}^{-1}$ we conclude that $I = A(\nu)$. Since the integrand is invariant under the above group H of translations we get the fundamental equation

$$A(\nu) = \sum_{\gamma} \int_{B_\gamma} g(x)\, dx + \int_{E_0} g(x)\, dx. \tag{8}$$

5. Estimate on the major arcs. Introduce a new variable $y = (y_1, \ldots, y_n)$ and set

$$\eta = (\omega_1 y_1 + \ldots + \omega_n y_n).$$

Let $Y(T)$ denote the domain in X where $0 < \eta < T$. Suppose furthermore that $(\alpha\gamma, \alpha_1\gamma, \ldots, \alpha_k\gamma) = \mathfrak{b}$ and let $\mathfrak{b}\mathfrak{d}$ have denominator $\mathfrak{a}$.

THEOREM 5.1. *If*

$$G(\gamma) = N(\mathfrak{a})^{-1} \sum_{\mu \bmod \mathfrak{a}} e(S(\phi(\mu)\,\gamma)),$$

then $G(\gamma) = O(N(\mathfrak{a})^{\epsilon - 1/k})$.

Proof. The proof of this result may be found in Hua [**5**].

THEOREM 5.2. *Let*

$$h(x) = \sum_{\lambda+\mu \in \mathscr{T}} e(S(\alpha(\lambda + \mu)^k \zeta)), \qquad \mathfrak{a}|\lambda,$$

where $\zeta = \xi - \gamma$, *then*

$$h(x) = N(\mathfrak{a})^{-1} \int_{Y(T)} e(S(\alpha(\eta + \mu)^k \zeta))\, dy + N(\mathfrak{a})^{-1} O(T^{n-a}).$$

Proof. The proof is almost identical with the corresponding result of Siegel; only the slightest modification is necessary.

THEOREM 5.3. *Let*

$$b(x) = \sum_{\lambda+\mu \in \mathscr{T}} e(S(\phi(\lambda + \mu)\zeta)), \qquad \mathfrak{a}|\lambda,$$

then

$$b(x) = h(x) + N(\mathfrak{a})^{-1} O(T^{n-a}).$$

Proof. Since $\lambda + \mu \in \mathscr{T}$, then

$$\phi(\lambda + \mu)\zeta - \alpha(\lambda + \mu)^k \zeta = \zeta\, O(T^{k-1}).$$

Therefore,

$$\begin{aligned} b(x) &= h(x) + O\Big(\sum_{\lambda+\mu\in\mathscr{T}} S(\phi(\lambda+\mu)\zeta - \alpha(\lambda+\mu)^k\zeta)\Big) && (\mathfrak{a}|\lambda) \\ &= h(x) + O\Big(\sum_{\lambda+\mu\in\mathscr{T}} S(|\zeta|\, T^{k-1})\Big) && (\mathfrak{a}|\lambda) \\ &= h(x) + O(T^{k+n-1})\, N(\mathfrak{a})^{-1}\, h^{-1}\, N(\mathfrak{a})^{-1/n} \\ &= h(x) + O(T^{n-\epsilon})\, N(\mathfrak{a})^{-1-1/n}. \end{aligned}$$

THEOREM 5.4.

$$f(x) = G(\gamma)\int_{Y(T)} e(S(\alpha\,\eta^k\zeta))\,dy + O(T^{n-\epsilon}).$$

Proof. We have

$$\begin{aligned} f(x) &= \sum_{\lambda\in\mathscr{T}} e(S(\phi(\lambda)(\zeta+\gamma))) \\ &= \sum_{\mu \bmod \mathfrak{a}} e(S(\phi(\mu)\gamma)) \sum_{\lambda+\mu\in\mathscr{T}} e(S(\phi(\lambda+\mu)\zeta)) && (\mathfrak{a}|\lambda) \\ &= \sum_{\mu \bmod \mathfrak{a}} e(S(\phi(\mu)\gamma))\Big\{N(\mathfrak{a})^{-1}\int_{Y(T)} e(S(\alpha\,\eta^k\zeta))\,dy + N(\mathfrak{a})^{-1}O(T^{n-\epsilon})\Big\} \\ &= N(\mathfrak{a})^{-1}\sum_{\mu \bmod \mathfrak{a}} e(S(\phi(\mu)\gamma))\int_{Y(T)} e(S(\alpha\,\eta^k\zeta))\,dy + O(T^{n-\epsilon}) \\ &= G(\gamma)\int_{Y(T)} e(S(\alpha\,\eta^k\zeta))\,dy + O(T^{n-\epsilon}). \end{aligned}$$

by Theorems 5.1, 5.2, and 5.3.

6. Estimate on the minor arc. We follow again in this section the procedure of Siegel [**10**; **11**] based on Weyl's method for estimating trigonometric sums. The presence of a polynomial in the exponent leads to no essential difficulty.

THEOREM 6.1. *Let*

$$\psi(\lambda) = S(\phi(\lambda)\xi), \quad \psi(\lambda;\lambda_1) = \psi(\lambda+\lambda_1) - \psi(\lambda),$$
$$\psi(\lambda;\lambda_1,\ldots,\lambda_m) = \psi(\lambda+\lambda_m;\lambda_1,\ldots,\lambda_{m-1}) - \psi(\lambda;\lambda_1,\ldots,\lambda_{m-1})$$

and A_m be the number of systems of integers $\lambda_1,\ldots,\lambda_m$ such that the 2^m simultaneous conditions

(9) $\lambda + \lambda_{p_1} + \ldots + \lambda_{p_g} \in \mathscr{T}\,(1 < p_1 < p_2 < \ldots p_g < m;\, g = 0,\ldots,m-1)$

have at least one solution λ. Then

$$|f(x)|^{2^m} < A_1^{2^{m-2}}\cdots A_{m-2}^2 A_{m-1} \sum_{\lambda_1,\ldots,\lambda_{m-1}} \Big|\sum_{\lambda} e(\psi(\lambda;\lambda_1,\ldots,\lambda_{m-1}))\Big|$$

for λ_i satisfying (9) *and $m = 1,\ldots,k-1$.*

Proof. The proof is by induction on m.

THEOREM 6.2.

$$|f(x)|^{2^{k-1}} = O(T^{n(2^{k-1}-k)}) \sum_{\lambda_1,\dots,\lambda_k} e(k!\, S(\alpha\lambda\lambda_1 \dots \lambda_{k-1}\xi)).$$

Proof. We first observe that $A_m = O(T^{nm})$; moreover

$$\sum_{m=1}^{k-1} m\, 2^{k-m-1} = 2^k - k - 1.$$

Since $\psi(\lambda; \lambda_1, \dots, \lambda_{k-1}) = S(k!\, \alpha\lambda\lambda_1 \dots \lambda_{k-1}\, \xi) + \psi(0; \lambda_1, \dots, \lambda_{k-1})$, the result follows.

THEOREM 6.3. *If x is a point of E_0, then*

$$f(x) = O(T^{n-(2^{k-1}+n)^{-1}+\epsilon}) = O(T^{n-\sigma+\epsilon}).$$

Proof. Let

$$\mu = \alpha k!\, \lambda_1 \dots \lambda_{k-1} \tag{11}$$

then we deduce

$$\begin{aligned} u &= \sum e(S(\lambda\mu\xi)) \\ &= \min(T, |e(S(\omega_1 \mu\xi)) - 1|^{-1}, \dots, |e(S(\omega_n\mu\xi)) - 1|^{-1})\, O(T^{n-1}). \end{aligned} \tag{12}$$

Let

$$S(\omega_j\, \alpha\mu\xi) = a_j + d_j \qquad (j = 1, \dots, n)$$

with rational integers a_j and $-\frac{1}{2} < d_j < \frac{1}{2}$, and define

$$\sum_{j=1}^{n} a_j\, \rho_j = \theta, \quad \sum_{j=1}^{n} d_j\, \rho_j = \tau.$$

We have $\theta \in \mathfrak{d}^{-1}$ and $e(S(\omega_j\alpha\mu\xi)) = e(d_j)$. Also $S(\omega_j\, \tau) = d_j$ and $\theta + \tau = \mu\xi$. Determine now numbers η and β with η an integer and β in $\mathfrak{d}^{-1}$ satisfying the condition of Theorem 3.2. There is an index $b \leqslant n$ such that $|\eta^{(b)}| > t$; let v denote the number of indices p satisfying $|\eta^{(p)}| < D^{-\frac{1}{2}}$, then $0 \leqslant v < n - 1$ and $p \neq b$. Let

$$q(\mu) = \min(T, |\tau^{(b)}|^{-1}), \tag{13}$$

then from (12), we deduce

$$u = O(T^{n-1}) \min(T, |\tau^{(b)}|^{-1}) = O(T^{n-1})\, q(\mu).$$

For given $\mu \neq 0$, the number of solutions of (11) subject to the condition $|\lambda_m| < 2T$ $(m = 1, \dots, k-1)$ is $O(T^{\epsilon})$ for arbitrarily small ϵ. On the other hand if $\mu = 0$, the number of solutions is $O(T^{n(k-2)})$. We conclude therefore

$$|f(x)|^{2^{k-1}} = O(T^{n(2^{k-1}-1)}) + O(T^{\epsilon+n(2^{k-1}-k+1)-1}) \sum_{\mu} q(\mu)$$

where μ runs over all integers satisfying

$$|\mu| < 2^k k!\, T^{k-1}. \tag{14}$$

With Siegel, we proceed to define $z_i = \tau^{(i)}$ and let $g_1, \dots, g_n$ be rational integers; $W = W(g_1, \dots, g_n)$ denotes the number of integers μ satisfying (14) and the further conditions

(15) $$g_i < 2D^{1/n} z_i \max(|\eta^{(i)}|, \ D^{-\frac{1}{2}}) < g_{i+1} \qquad (i = 1, \dots, n).$$

Let μ_0 be a fixed one of these μ and set $\mu_0\xi = \theta_0 + \tau_0$, $\eta\xi - \beta = \delta$. We have

$$\delta(\mu - \mu_0) - \eta(\tau - \tau_0) = \eta(\theta - \theta_0) - \beta(\mu - \mu_0) = \kappa.$$

By observing that κ lies in $\mathfrak{d}^{-1}$, we deduce that $\kappa = 0$. It follows therefore that $\eta|\beta\mathfrak{d}(\mu - \mu_0)$, and since $N((\eta, \beta\mathfrak{d})) < D^{\frac{1}{2}}$ then $\eta|c(\mu - \mu_0)$ where c is a positive rational integer depending only on the field F. It follows that

$$(\mu - \mu_0)\,\eta^{-1} = \eta^{-1}\,O(T^{k-1})$$

and hence that

$$(\mu^{(p)} - \mu_0^{(p)})\,\eta^{(p)-1} = (\tau^{(p)} - \tau_0^{(p)})\,\delta^{(p)-1} = O(h).$$

Consequently the number of differences $\mu - \mu_0$ is

$$1 + O(h^n) \prod (|\eta^{(i)}|^{-1}\, T^{k-1}), \qquad i \neq p.$$

Therefore

(16) $$W = O(1) + O(T^{n(k-1)+an}) \prod |\eta^{(i)}|^{-1}, \qquad i \neq p.$$

If $W > 0$, then $g_p = O(1)$ and $g_i = O(\eta^{(i)})$ if $i \neq p$. If then g_b is fixed, the number of systems $g_1, \dots, g_n$ with $W > 0$ is

$$O(\prod \eta^{(i)}), \qquad i \neq p, b.$$

Then the number of integers μ in F satisfying (14) and the single condition

(17) $$g < 2D^{1/n} z_b |\eta^{(b)}| < g + 1$$

has value

$$W_g = \sum_{g_i} W(g_1, \dots, g_n) = T^{(n-1)(k+a-1)}\, O(1 + T^{k-1}|\eta^{(b)}|^{-1}), \qquad i \neq b.$$

On the other hand,

$$L = \sum_{0<g<|\eta^{(b)}|} \min(T, g^{-1}\eta^{(b)}) = T^{k+a-1}\, O(\log T).$$

Furthermore, $LT^{k-1}|\eta^{(b)}|^{-1} = O(T^{k+a-1})$, therefore

$$\begin{aligned}\sum_{\mu} q(\mu) &= \sum_{g} W_g\, O(\min(T, |g|^{-1}|\eta^{(b)}|, |g+1|^{-1}|\eta^{(b)}|)) \\ &= L\, T^{(n-1)(k+a-1)}\, O(1 + T^{k-1}|\eta^{(b)}|^{-1}) \\ &= O(T^{n(k+a-1)}).\end{aligned}$$

The estimate for $f(x)$ now follows.

7. Further estimates.

THEOREM 7.1. *Let*

$$\Phi(\xi) = \Phi_T(\xi) = \left(\int_{Y(T)} e(S(\alpha\eta^k\xi))\, dy \right)^s ;$$

then

$$\sum_\gamma \int_{B_\gamma} g(x)\, dx = \sum_\gamma G^s(\gamma)\, e(-S(\nu\gamma)) \int_B \Phi(\zeta)\, e(-S(\nu\zeta))\, dx + O(T^{n(s-k)}).$$

Proof. Let $u_i = T^{-1}\eta^{(i)}$ $(i = 1, \ldots, n)$ and $T^k\zeta = \tau$. Then

$$\int_{Y(T)} e(S(\alpha\eta^k\zeta))\, dy = D^{-\frac{1}{2}}\, T^n N\left(\int_0^1 e(\tau^{(i)}\alpha^{(i)} u^k)\, du \right). \tag{18}$$

Also

$$\int_0^1 e(\alpha^{(i)}\tau^{(i)} u^k)\, du = O(\min(1, |\tau^{(i)}|^{-1/k})) \tag{19}$$

and if x is a point of B_γ, then by Theorem 5.4,

$$\begin{aligned} f^s(x) - \left(G(\gamma) \int_{Y(T)} e(S(\alpha\eta^k\zeta))\, dy \right)^s \\ = O(T^{n-a}) \max \left(f(x), G(\gamma) \int_{Y(T)} e(S(\alpha\eta^k\zeta))\, dy \right)^{s-1}. \end{aligned} \tag{20}$$

By (19) and Theorem 5.1, we obtain

$$\begin{aligned} f^s(x) = G^s(\gamma) \left(\int_{Y(T)} e(\alpha\eta^k\zeta)\, dy \right)^s \\ + O(T^{ns-a})\, N(\mathfrak{a}^{\epsilon-(s-1)/k})\, N(\min(1, |\tau^{-1/k}|)^{s-1}). \end{aligned} \tag{21}$$

On the other hand,

$$\int_{B_\gamma} N(\min(1, |\tau^{-1/k}|)^{s-1})\, dx = O(T^{-kn}), \tag{22}$$

and since by partial summation $\sum_\gamma N(\mathfrak{a})^{-(s-1)/k} = O(1)$, we have

$$\begin{aligned} \sum_\gamma \int_{B_\gamma} g(x)\, dx &= \sum_\gamma \int_{B_\gamma} f^s(x)\, e(-S(\nu\xi))\, dx \\ &= \sum_\gamma G^s(\gamma)\, e(-S(\nu\gamma)) \int_{B_\gamma} \Phi(\zeta)\, e(-S(\nu\zeta))\, dx \\ &\qquad + \sum N(\mathfrak{a})^{-(s-1)/k} O(T^{sn-a})\, O(T^{-kn}) \\ &= \sum_\gamma G^s(\gamma)\, e(-S(\nu\gamma)) \int_{B_\gamma} \Phi(\zeta)\, e(-S(\nu\zeta))\, dx + o(T^{n(s-k)}). \end{aligned}$$

THEOREM 7.2.

$$\sum_\gamma \int_{B_\gamma} g(x)\, dx = \sum_\gamma G^s(\gamma)\, e(-S(\nu\gamma)) \int_X \Phi(\xi)\, e(-S(\nu\xi))\, dx + o(T^{n(s-k)}).$$

Proof. We replace here, it will be noted, B_γ by X. It is therefore sufficient to prove that

$$U = \sum_\gamma G^s(\gamma)\, e(-S(\nu\gamma)) \int_{X-B_\gamma} \Phi(\zeta)\, e(-S(\nu\zeta))\, dx = o(T^{n(s-k)}).$$

If x is a point of $X - B_\gamma$, then by (5) there is at least one index i such that $h|\zeta^{(i)}| > N(\mathfrak{a})^{-1/n}$. Therefore,

$$\begin{aligned}\int_{X-B_\gamma} \Phi(\zeta)\, e(-S(\nu\zeta))\, dx &= O(T^{ns}) \int_{X-B_\gamma} N(\min(1, \tau^{-s/k}))\, dx \\ &= O(T^{n(s-k)}) \int_{x_1 > tN(\mathfrak{a})^{-1/n}} x_1^{-s/k}\, dx_1 \\ &= O(T^{n(s-k)})(t\, N(\mathfrak{a})^{-1/n})^{1-s/k}.\end{aligned}$$

Consequently,

$$\begin{aligned}U &= O(T^{n(s-k)}) \sum_\gamma |G(\gamma)|^s\, t^{1-s/k} N(\mathfrak{a})^{-(1-s/k)/n} \\ &= O(T^{n(s-k)})\, t^{1-s/k} \sum_\gamma N(\mathfrak{a})^{-s/k-1/n+s/nk} \\ &= O(T^{n(s-k)})\, t^{1+n-s/k} \\ &= o(T^{n(s-k)}).\end{aligned}$$

THEOREM 7.3.

$$\int_E |f(x)|^{2^k}\, dx = O(T^{n(2^k-k)+\epsilon}).$$

Proof. This theorem was proved, for the rational field by Hua [6]; this is an extension to the present case. The proof proceeds by induction on k. For $k = 0$ the result is trivial, assume it true for $k - 1$. Then

$$\begin{aligned}\int_E |f(x)|^{2^k}\, dx &= \int_E |f(x)|^{2^{k-1}} |f(x)|^{2^{k-1}}\, dx \\ &= \int_E |f(x)|^{2^{k-1}} \Big\{ O(T^{n(2^{k-1}-1)}) \\ &\quad + O(T^{n(2^{k-1}-k)}) \sum_{\lambda_1} \cdots \sum_{\lambda_{k-1}} \sum_\lambda{}^{*}\, e(S(k!\, \alpha\lambda_1 \ldots \lambda_{k-1}\lambda\xi)) \Big\}\, dx,\end{aligned}$$

by Theorem 6.2, the asterisk indicating that the summation excludes the value 0 of $\lambda_1, \ldots, \lambda_{k-1}, \lambda$.

By the inductive hypothesis however, we have

$$\begin{aligned}\int_E |f(x)|^{2^k}\, dx &= O(T^{n(2^{k-1}-1)}) T^{n(2^{k-1}-k+1)+\epsilon} \\ &\quad + O(T^{n(2^{k-1}-k)}) \int_E |f(x)|^{2^{k-1}} \sum_{\lambda_1} \cdots \sum_{\lambda_{k-1}} \sum_\lambda e(S(k!\, \alpha\lambda_1 \ldots \lambda_{k-1}\xi))\, dx \\ &\doteq O(T^{n(2^{k-1}-k)+\epsilon}) - O(T^{n(2^{k-1}-k)}) \\ &\quad \cdot \int_E \Big\{ \sum_{\mu_1} \cdots \sum_{\mu_{2^{k-1}}} e(S(\phi(\mu))) \sum_{\lambda_1} \cdots \sum_{\lambda_{k-1}} \sum_\lambda{}^{*} e(S(\theta)) \Big\}\, dx\end{aligned}$$

where

$$\phi(\mu) = (\phi(\mu_1) - \phi(\mu_2) + \ldots - \phi(\mu_{2^k-1}))\xi$$

and

$$\theta = k!\,\alpha\lambda_1 \ldots \lambda_{k-1}\lambda\xi.$$

This follows by writing the square of the absolute value as the product of complex conjugates and noting therefore that

$$|f(x)|^{2^{k-1}} = \sum_{\mu_1} \ldots \sum_{\mu_{2^{k-1}}} e(S((\phi(\mu_1) - \phi(\mu_2) + \ldots - \phi(\mu_{2^k-1}))\xi))).$$

Therefore, using the properties of $\mathfrak{d}^{-1}$, we get

$$\int_E |f(x)|^{2^k} = O(T^{n(2^k-k)+\epsilon}) - O(T^{n(2^{k-1}-k)}).\ C,$$

where C is the number of solutions of the equation $\phi(\mu) = \theta$, the λ and μ being restricted by the conditions $|\lambda_i| < T$, $|\mu_i| < T$.

On the other hand, as in the rational case, it is proved that

$$C = O(T^{\epsilon}\, T^{n(2^{k-1})}).$$

We conclude therefore finally,

$$\int_E |f(x)|^{2^k} = O(T^{n(2^k-k)+\epsilon}) - O(T^{n(2^{k-1}-k)+n2^{k-1}+\epsilon})$$
$$= O(T^{n(2^k-k)+\epsilon}).$$

8. Proof of the asymptotic formula for $A(\nu)$. In the same way as in Siegel **[11]**, we can prove the following

THEOREM 8.1.

$$I = \int_X e(-S(\nu\xi))\,\Phi_T(\xi)\,dx = D^{\frac{1}{2}(1-s)}\left(\frac{\Gamma^s(1+1/k)}{\Gamma(s/k)}\right)^n N(\alpha)^{-s/k}\,N(\nu)^{-1+s/k},$$

Again using Dirichlet's theorem on units, we could prove

THEOREM 8.2. *Let θ_0 be an algebraic integer, then there exists a totally positive unit η such that $\theta = \eta^k\theta_0$ fulfils the conditions*

$$c_1 N(\theta)^{1/n} < \theta < c_2 N(\theta)^{1/n}$$

with c_1 and c_2 real numbers.

We now show that the "singular series" converges.

THEOREM 8.3. *If γ runs over a complete system of modulo $\mathfrak{d}^{-1}$ incongruent numbers in F, then the "singular series"*

$$\sigma' = \sigma'_{\nu,k,s} = \sum_{\gamma} G^s(\gamma)\,e(-S(\nu\gamma))$$

is convergent for $s > 2k + 1$.

Proof. Suppose

$$H(\mathfrak{a}) = \sum G^s(\gamma)\, e(-S(\nu\gamma))$$

the summation being over a complete system of modulo $(\mathfrak{a}\mathfrak{d})^{-1}$ incongruent numbers γ such that the denominator of $\gamma\mathfrak{d}$ is $\mathfrak{a}$. Then

$$\sigma' = \sum_{\mathfrak{a}} \sum_{\gamma \bmod (\mathfrak{a}\mathfrak{d})^{-1}} G^s(\gamma)\, e(-S(\nu\gamma)) = \sum_{\mathfrak{a}} H(\mathfrak{a}).$$

Therefore by Theorem 5.1,

$$\sigma' = O(1) \sum_{\mathfrak{a}} |H(\mathfrak{a})| = O(1) \sum_{\mathfrak{a}} N(\mathfrak{a})^{1+\epsilon-s/k} = O(1).$$

COROLLARY.

$$\sigma' = \sum_{\gamma\epsilon\Gamma} G^s(\gamma)\, e(-S(\nu\gamma)) + o(1).$$

The proof of our main theorem is now merely a collection of the results established. It is clear that $A(\nu) = A(\nu\eta^k)$ where η is a unit. Put $N(\nu)^{1/n} = T^k$; then by Theorem 8.2, we may assume that

$$c_1 N(\nu)^{1/n} < \nu < c_2 N(\nu)^{1/n}.$$

By (8), we have

$$A(\nu) = \sum_{\gamma} \int_{B_\gamma} g(x)\, dx + \int_{E_0} g(x)\, dx.$$

By Theorems 7.2 and 8.1, we have,

$$\sum_{\gamma} \int_{B_\gamma} g(x)\, dx = \sigma' I + o(N(\nu)^{-1+s/k}).$$

On the other hand, using Theorems 6.2, and 7.3, we deduce

$$\begin{aligned} \int_{E_0} g(x) &= O(1) \int_{E_0} |f(x)|^s\, dx \\ &= O(1) \int_{E_0} |f(x)|^{s-2^k+2^k}\, dx \\ &= O(T^{(n-(2^{k-1}+n)^{-1}+\epsilon)(s-2^k)}) \int_{E} |f(x)|^{2^k}\, dx \\ &= O(T^{(n-(2^{k-1}+n)^{-1}+\epsilon)(s-2^k)})\, O(T^{n(2^k-k)+\epsilon}) \\ &= o(N(\nu)^{-1+s/k}). \end{aligned}$$

This completes the proof.

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ON INTEGERS n RELATIVELY PRIME TO $[\alpha n]$

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1. Introduction. The object of this paper is to consider a problem suggested by Dr. K. F. Roth, on the distribution of integers n that are relatively prime to the integral part of αn, α being a fixed real number. He conjectured that the number of positive integers up to x with this property is asymptotic to $6x/\pi^2$ (or in other words that they have the density $6/\pi^2$), for irrational α. I prove this and rather more in the following

THEOREM. *For every real number α the positive integers n such that*

$$(n, [\alpha n]) = 1 \tag{1}$$

have a density $\delta(\alpha)$. For every irrational α, $\delta(\alpha) = 6/\pi^2$. For rational $\alpha = a/q$, with $(a, q) = 1$ and $q > 0$, $\delta(\alpha)$ depends only on q and has the value

$$q^{-1}\sum_{u=1}^{q-1} u^{-1}\,\phi(u),$$

which tends to the limit $6/\pi^2$ as $q \to \infty$.

Notation. Throughout the paper, Greek letters denote real numbers, ϵ being positive and arbitrarily small. Latin letters denote rational integers, n, q, q', x, d, R being positive, and a, q coprime. $\phi(x)$ and $\mu(x)$ are the functions of Euler and Möbius, $d(x)$ is the number of divisors of x, and (y, z) is the highest common factor of y and z (not both zero). $[\alpha]$ is the greatest integer not exceeding α. The constants implied by the O-notation are absolute, except in formulae containing ϵ, in which they depend on ϵ only.

We define

$$f(x, \alpha) = \sum_{n\leq x,\,(1)} 1,$$

where (1) refers to equation (1) above, and

$$\psi(q, r) = \sum_{-r\leq u<q-r} |u|^{-1}\,\phi(|u|), \qquad u \neq 0$$

where an empty sum is to be interpreted as zero. Thus $\delta(\alpha)$ is the limit (to be proved to exist) of $x^{-1}f(x, \alpha)$ as $x \to \infty$, and we have to show that $\delta(a/q) = \psi(q, 0)$.

2. Preliminary. The first of the following lemmas is a known result due to Vinogradov [**1**, chap. II, Ex. 19b], but the proof is reproduced as it is short.

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LEMMA 1. *For any l, u, x with $u \neq 0$ we have*

$$\sum_{n \leq x,\, (n+l,u)=1} 1 = x|u|^{-1}\,\phi(|u|) + O(d(|u|)).$$

Proof. We may suppose $u > 0$, and $0 \leq l < u$. Then the result for positive l follows from that for $l = 0$ by putting first $x + l$ and then l for x, and subtracting; so we may suppose $l = 0$. With these preliminaries the sum to be estimated is equal to

$$\begin{aligned}
\sum_{n=1}^{x} \sum_{d|u,\, d|n} \mu(d) &= \sum_{d|u} \sum_{n \leq x,\, d|n} \mu(d) \\
&= \sum_{d|u} \left[\frac{x}{d}\right] \mu(d) \\
&= \sum_{d|u} \frac{x}{d}\,\mu(d) + O(d(u)) \\
&= xu^{-1}\,\phi(u) + O(d(u)).
\end{aligned}$$

LEMMA 2.

$$\psi(q, r) = \frac{6q}{\pi^2} + O(q^{\epsilon}) + O(|r|^{\epsilon}).$$

Proof. It is sufficient to consider the case $r = 0$, which follows by partial summation from the known result [**2**, p. 266, Theorem 330]

$$\sum_{n=1}^{x} \phi(n) = \frac{3x^2}{\pi^2} + O(x \log x).$$

LEMMA 3. *For $R \geq q^2$,*

$$R^{-1} \sum_{r=0}^{R-1} \psi(q, r) = \frac{6q}{\pi^2} + O(1). \tag{2}$$

and

$$R^{-1} \sum_{r=-R}^{-1} \psi(q, r) = \frac{6q}{\pi^2} + O(1). \tag{3}$$

Proof. We count the number of times a summand with given u occurs in the double sum obtained by substituting the sum defining $\psi(q, r)$ in that on the left of (3); with unimportant exceptions it is precisely q. Thus

$$\begin{aligned}
\sum_{r=-R}^{-1} \psi(q, r) &= \sum_{r=-R}^{-1} \sum_{-r \leq u < q-r,} |u|^{-1}\,\phi(|u|) \qquad u \neq 0 \\
&= q \sum_{1 \leq u < R} u^{-1}\phi(u) - \sum_{u=1}^{q-1} (q-u)\, u^{-1}\phi(u) \\
&\qquad + \sum_{v=1}^{q} (q - v + 1)(R + v - 1)^{-1}\phi(R + v - 1) \\
&= q\psi(R, 0) + O(q^2).
\end{aligned}$$

Hence, using $R > q^2$, the left member of (3) is

$$qR^{-1}\psi(R, 0) + O(1),$$

and (3) follows on putting R, 0, $\frac{1}{2}$ for q, r, ϵ in Lemma 2.

The proof of (2) is similar.

3. The case of rational α. In this section we take $\alpha = a/q$, and prove a lemma which is a slight generalization of the latter part of the Theorem.

LEMMA 4. *We consider n satisfying (for fixed a, q, r)*

$$\left(n, \left[\frac{an + r}{q}\right]\right) = 1, \tag{4}$$

and define

$$F(x; a, q, r) = \sum_{n<x, (4)} 1.$$

Then (interpreting 0 log 0 *as* 0) *we have*

$$F(x; a, q, r) = xq^{-1}\psi(q, r) + O(q \log q) + O(|r| \log |r|) + O(1).$$

Proof We write

$$n = qm + l, \qquad 0 \le l < q, \tag{5}$$

$$\left[\frac{an + r}{q}\right] = am + l'. \tag{6}$$

We can choose a', q' so that $a'q - aq' = \pm 1$, whence

$$(y, z) = (ay - qz, a'y - q'z),$$

and so

$$\left(n, \left[\frac{an + r}{q}\right]\right) = (u, m + l'')$$

where l'' is independent of m and

$$u = al - ql'. \tag{7}$$

It is clear from (5) to (7) that $-r \le u < q - r$ and that u runs with l through a complete set of residues modulo q. Hence for $l = 0, 1, \ldots, q - 1$, u takes the values $-r, -r + 1, \ldots, q - r - 1$, in some order, each just once.

Now we break up the sum $F(x; a, q, r)$ into a double sum over l, m, or equivalently, over u, m. For $u = 0$, the inner sum over m is $O(1)$, since (4) can hold only if $m = -l'' \pm 1$. For other u, we have to sum over $m = 0, 1, \ldots, [x/q]$, and possibly $[x/q] + 1$. But with error $O(1)$ we can omit the values 0, $[x/q] + 1$. Then using Lemmas 1 and 2 we find (for $u \ne 0$)

$$F(x; a, q, r) = \sum_{-r \le u < q-r} \sum_{\substack{1 \le m \le [x/q], \\ (u, m+l'')=1}} 1 + O(q)$$

$$= \sum_{-r<u<q-r} \left[\frac{x}{q}\right] |u|^{-1}\phi(|u|) + O\Big(\sum_{-r<u<q-r} d(|u|)\Big) + O(q)$$
$$= xq^{-1}\,\psi(q, r) + O\Big(\sum_{-r<u<q-r} d(|u|)\Big) + O(q).$$

The Lemma now follows from

$$d(1) + d(2) + \ldots + d(x) = O(x \log x);$$

more precise results implying this are well known [**2**, p. 262, Theorems 318 to 320].

4. Proof of the Theorem. For rational α, we have only to take $r = 0$ in Lemmas 2 and 4.

Now let α be irrational, and let a/q, a'/q' be two successive convergents to its infinite continued fraction expansion. (In the case of negative α, which we could of course avoid, the convergents are those of the continued fraction for $|\alpha|$, with the signs of the numerators changed.) For large x, we choose q to satisfy

$$q \leqslant x(\log x)^{-2} < q'. \tag{8}$$

Clearly q tends to infinity with x, and the theorem follows if we prove

$$f(x, \alpha) = \frac{6x}{\pi^2} + O(x/\log x) + O(xq^{-\frac{1}{2}}). \tag{9}$$

We define $r = r(n) = r(n, \alpha, q)$ by

$$r = [n(q\alpha - a)], \tag{10}$$

whence

$$[\alpha n] = \left[\frac{an + r}{q}\right]. \tag{11}$$

As n takes the values $1, 2, \ldots, x$, we note that r takes the values $0, 1, \ldots, R - 1$ or $-1, -2, \ldots, -R$, according to the sign of $\alpha - a/q$, where, by (8) and since $|\alpha - a/q| < 1/qq'$, we have

$$R < 1 + x/q' < 1 + \log^2 x. \tag{12}$$

If $q > \log^3 x$, (11) and (12) show that $[\alpha n] = [an/q]$ except possibly for n in $O(R) = O(q/\log x)$ residue classes (mod q). Now by (8) there are, up to x, only $O(x/\log x)$ such n, so (9) follows from

$$\begin{aligned} f(x, \alpha) &= f(x, a/q) + O(x/\log x) \\ &= F(x; a, q, 0) + O(x/\log x) \\ &= xq^{-1}\,\psi(q, 0) + O(q \log q) + O(x/\log x) \\ &= xq^{-1}\,\psi(q, 0) + O(x/\log x), \end{aligned}$$

using Lemmas 2 and 4 (with $r = 0$) and (8).

We may therefore assume

$$q \leqslant \log^3 x. \tag{13}$$

We write

$$f(x, \alpha) = \sum_r \sum_{X_r < n \leqslant Y_r, (1)} 1,$$

where $X_r + 1, \ldots, Y_r$ $(= X_{r\pm1}$ unless $r = R - 1$ or $-R)$ are the consecutive values of n for which r takes a given value. The outer sum is over $0 \leqslant r < R$ or $0 > r \geqslant -R$ as the case may be. By (11) and Lemma 4, the inner sum is

$$F(Y_r; a, q, r) - F(X_r; a, q, r) = xq^{-1}\psi(q, r) + O(q \log q) + O(R \log R).$$

Hence using (12) and (13) we find

$$f(x, \alpha) = \sum_r (Y_r - X_r)\, q^{-1}\, \psi(q, r) + O(\log^6 x). \tag{14}$$

Now we may assume

$$R \geqslant q^2. \tag{15}$$

For otherwise (9) follows immediately from (14) and Lemma 2.

We next note that, except for $r = R - 1$ or $-R$, $Y_r - X_r$ can take only two different values; these are consecutive integers, the smaller of which is $[|q\alpha - a|^{-1}]$. It easily follows that

$$Y_r - X_r = \begin{cases} xR^{-1} + O(xR^{-1}), & \text{if } r = R - 1 \text{ or } -R, \\ xR^{-1} + O(xR^{-2}), & \text{otherwise.} \end{cases}$$

Substituting in (14) we find

$$f(x, \alpha) = xq^{-1} R^{-1} \sum_r \psi(q, r) + O(xR^{-1}) + O(\log^6 x).$$

Now (9) follows from (15) and Lemma 3, and so the proof is complete.

5. Conclusion. It would not be difficult to prove a similar result (with $6/\pi^2 k^2$ in place of $6/\pi^2$) for n satisfying $(n, [\alpha n]) = k$ in place of (1).

I am indebted to Professor Davenport and Dr. Roth for reading earlier drafts of this paper, and pointing out some obscurities and errors.

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ON THE NUMBER OF PRIMITIVE LATTICE POINTS IN A PARALLELOGRAM

THEODOR ESTERMANN

1. Let α be any irrational real number, and let $F(u)$ denote the number of those positive integers $n < u$ for which $(n, [n\alpha]) = 1$. Watson proved in the preceding paper that

(1) $$\lim_{u \to \infty} \{u^{-1} F(u)\} = 6\pi^{-2}.$$

The object of this paper is to give a different proof of a slight generalization of this result.

In what follows, a lattice point is a point in the plane whose cartesian coordinates are integers. It is said to be primitive if its coordinates are relatively prime. For any positive numbers u and a, let $g(u, a)$ denote the number of lattice points, $f(u, a)$ the number of primitive lattice points in the set of points given by

$$0 < x < u, \qquad \alpha x - a < y < \alpha x$$

(a parallelogram with two of its sides included). Then $F(u) = f(u, 1)$, and thus the formula

(2) $$\lim_{u \to \infty} \{u^{-1} f(u, a)\} = 6\pi^{-2} a$$

is a generalization of (1).

2. My proof is based on the formula

(3) $$\lim_{u \to \infty} \{u^{-1} g(u, a)\} = a.$$

This is equivalent to a well-known theorem of Bohl, Sierpinski, and Weyl [**2**, Satz 2]. The following simple elementary proof of (3) is reconstructed from what I remember of a lecture given by Hecke about thirty years ago. I cannot trace it in the literature, but its main idea, at any rate, is due to Hecke.

Since the addition of an integer to α does not alter $g(u, a)$, we may assume that $\alpha > 0$. Then, by a theorem of Kronecker [**1**, Theorem 438], the numbers of the form $m\alpha - n$, where m and n are positive integers, are everywhere dense. It is therefore sufficient to consider the case when a is of this form. Then

(4) $$g(u, a) = g(u, m\alpha - n) = g(u, m\alpha) - n[u],$$

and $g(u, m\alpha) = A + B - C$, where A, B, and C are, respectively, the numbers

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of lattice points in the parallelogram

$$0 < y \leqslant \alpha u, \quad y/\alpha \leqslant x < y/\alpha + m,$$

in the triangle

$$0 < x, \quad \alpha x - m\alpha < y \leqslant 0$$

and in the triangle

$$u < x, \quad \alpha x - m\alpha < y \leqslant \alpha u.$$

Now $A = m[\alpha u]$, B is independent of u, and C is a bounded function of u. Hence

$$\lim_{u \to \infty} \{u^{-1} g(u, m\alpha)\} = m\alpha,$$

and, by (4),

$$\lim_{u \to \infty} \{u^{-1} g(u, a)\} = m\alpha - n = a.$$

3. Throughout the remainder of this paper, let the letters d, k, and n denote positive integers, m an integer, a and u positive numbers, and μ Möbius' function. Then

$$\begin{aligned} (5) \qquad f(u, a) &= \sum_{n \leqslant u} \sum_{\substack{m \\ an-a<m\leqslant an \\ (n,m)=1}} 1 = \sum_{n \leqslant u} \sum_{\substack{m \\ an-a<m\leqslant an}} \sum_{d \mid (n,m)} \mu(d) \\ &= \sum_d \mu(d) \sum_{\substack{n \leqslant u \\ d \mid n}} \sum_{\substack{m \\ an-a<m\leqslant an \\ d \mid m}} 1 = \sum_d \mu(d) \sum_{n' \leqslant u/d} \sum_{\substack{m' \\ an'-a/d<m'\leqslant an'}} 1 \\ &= \sum_d \mu(d)\, g(u/d, a/d), \end{aligned}$$

so that

$$(6) \qquad u^{-1} f(u, a) = \sum_d \mu(d)\, d^{-1} (u/d)^{-1} g(u/d, a/d).$$

Also, by (3),

$$(7) \qquad \lim_{u \to \infty} \{(u/d)^{-1} g(u/d, a/d)\} = a/d.$$

Thus, if it were permissible to proceed to the limit term by term, it would follow from (6) that

$$\lim_{u \to \infty} \{u^{-1} f(u, a)\} = a \sum_d \mu(d)\, d^{-2} = 6\pi^{-2} a;$$

but I do not know any direct method of justifying this process.

4. A slight modification of the preceding argument, however, will lead to

$$(8) \qquad \overline{\lim_{u \to \infty}} \{u^{-1} f(u, a)\} \leqslant 6\pi^{-2} a.$$

Let

$$(9) \qquad h(u, a, k) = \sum_{n \leqslant u} \sum_{\substack{m \\ an-a<m\leqslant an \\ (n,m,k!)=1}} 1.$$

Then, by the first equation in (5),

$$f(u, a) \leq h(u, a, k) \tag{10}$$

and, by the argument that led to (6),

$$u^{-1}\, h(u, a, k) = \sum_{d \mid k^!} \mu(d)\, d^{-1} (u/d)^{-1}\, g(u/d, a/d).$$

Here it is obviously permissible to proceed to the limit term by term. From this and (7) it follows that

$$\lim_{u \to \infty} \{u^{-1}\, h(u, a, k)\} = a \sum_{d \mid k^!} \mu(d)\, d^{-2}. \tag{11}$$

By (10) and (11),

$$\overline{\lim_{u \to \infty}}\, \{u^{-1} f(u, a)\} \leq a \sum_{d \mid k^!} \mu(d)\, d^{-2}.$$

Since this holds for every k, and

$$\lim_{k \to \infty} \sum_{d \mid k^!} \mu(d)\, d^{-2} = \sum_{d} \mu(d)\, d^{-2} = 6\pi^{-2},$$

we deduce (8).

5. To complete the proof of (2), we note that, by (9),

$$\begin{aligned} h(u, a, k) &= \sum_{\substack{d \\ (d, k^!) = 1}} \sum_{n \leq u} \sum_{\substack{m \\ an - a < m \leq an \\ (n, m) = d}} 1 \\ &= \sum_{\substack{d \\ (d, k^!) = 1}} \sum_{n' \leq u/d} \sum_{\substack{m' \\ an' - a/d < m' \leq an' \\ (n', m') = 1}} 1 = \sum_{\substack{d \\ (d, k^!) = 1}} f(u/d, a/d). \end{aligned}$$

Hence

$$f(u, a) = h(u, a, k) - \sum_{\substack{d > 1 \\ (d, k^!) = 1}} f(u/d, a/d). \tag{12}$$

LEMMA 1. $f(u, a) - f(\tfrac{1}{2}u, a) \leq 2au + 1$.

Proof.

$$\begin{aligned} f(u, a) - f(\tfrac{1}{2}u, a) &= \sum_{\substack{n \\ \frac{1}{2}u < n \leq u}} \sum_{\substack{m \\ an - a < m \leq an \\ (n, m) = 1}} 1 \\ &\leq \sum_{n \leq u} \sum_{\substack{m \\ a - 2a/u < m/n \leq a \\ (n, m) = 1}} 1. \end{aligned}$$

This is the number of fractions, in their lowest terms, with positive denominators less than or equal to u, in an interval of length $2a/u$. Since any two such fractions differ by at least u^{-2}, the result follows.

LEMMA 2. *Let* $u \geq 1$. *Then* $f(u, a) \leq 4au + \log(2u)/\log 2$.

Proof. Let $b = [\log u/\log 2]$. Then, by Lemma 1,

$$f(u, a) = f(u, a) - f(2^{-b-1}u, a) = \sum_{m=0}^{b} \{f(2^{-m}u, a) - f(2^{-m-1}u, a)\}$$
$$< \sum_{m=0}^{b} (2^{1-m}au + 1) < 4au + b + 1,$$

and the result follows.

LEMMA 3. *Let $au \leqslant 1$. Then $f(u, a) \leqslant 1$.*

Proof. Otherwise there would be two distinct fractions m_1/n_1 and m_2/n_2, such that

$$n_1 \leqslant n_2 \leqslant u, \quad \alpha - a/n_1 < m_1/n_1 \leqslant \alpha,$$
$$\alpha - a/n_1 \leqslant \alpha - a/n_2 < m_2/n_2 \leqslant \alpha,$$

which implies that $|m_1/n_1 - m_2/n_2| < a/n_1$; but

$$|m_1/n_1 - m_2/n_2| \geqslant 1/(n_1 n_2) \geqslant 1/(n_1 u) \geqslant a/n_1.$$

6. Let $u \geqslant 1$. Then, since the conditions $d > 1$ and $(d, k!) = 1$ imply that $d > k$, and since $f(u/d, a/d) = 0$ if $d > u$, it follows from Lemmas 2 and 3 that

$$\sum_{\substack{d \\ 1<d\leqslant\sqrt{(au)} \\ (d,k!)=1}} f(u/d, a/d) \leqslant \sum_{\substack{d \\ k<d\leqslant\sqrt{(au)}}} (4aud^{-2} + 2\log u) \leqslant 4auk^{-1} + 2\sqrt{(au)}\log u$$

and

$$\sum_{\substack{d>\sqrt{(au)} \\ (d,k!)=1}} f(u/d, a/d) \leqslant \sum_{\substack{d\leqslant u \\ (d,k!)=1}} 1 .$$

Hence, by (12),

$$u^{-1}f(u, a) \geqslant u^{-1}h(u, a, k) - \frac{4a}{k} - 2\sqrt{(a/u)}\log u - \frac{1}{u}\sum_{\substack{d\leqslant u \\ (d,k!)=1}} 1 ,$$

and hence, by (11),

$$\varliminf_{u\to\infty} \{u^{-1}f(u, a)\} \geqslant a\sum_{d|k!} \mu(d)\, d^{-2} - \frac{4a}{k} - \frac{\phi(k!)}{k!},$$

where ϕ denotes Euler's function. Since this holds for every k, and the right-hand side tends to $6\pi^{-2}a$ as $k \to \infty$, it follows that

$$\varliminf_{u\to\infty} \{u^{-1}f(u, a)\} \geqslant 6\pi^{-2}a,$$

which, together with (8), proves (2).

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POSTULATES FOR BOOLEAN ALGEBRAS

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Introduction. The independence of postulates for well-known systems is a question of general interest. A closely related question is whether or not, by altering one or more of the postulates in an independent set, the set remains independent. From this standpoint the best sets of postulates are those which involve, first, the fewest postulates and, next, the least number of variables. As a rule progress is made in this direction at the sacrifice of simplicity of postulates. In this paper, in counting postulates, we ignore properties such as closure under the operations and count only identities or those stating that one equation implies another.

For Boolean algebras, sets of three postulates in three variables are common. We mention here only three particularly simple sets. The author's set [**11**] is of interest because two of the three postulates describe distributive lattices. Byrne [**5**] has an elegant set which consists of only two postulates if the postulate in the form of a double implication is not counted twice. A very simple set may be derived from Set IV of Huntington [**10**] by replacing commutativity and associativity by cyclic associativity (compare [**5**]).

Sets of two postulates are less common. Croisot [**7**] has one in five variables based on the ternary median operation and complements. Bernstein [**2**] has one in four variables based on the operation of implication. Bernstein [**1**] has a set in only three variables based on the stroke operation of Sheffer. Sets of two postulates based on ring operations have been given by Bernstein [**3**] and Byrne [**6**] but these involve seven and nine variables. In §§1 and 2 below we show that the number of variables may be reduced to four. This is made possible by the introduction of a single postulate for Boolean groups.

The only single postulate system for Boolean algebras that has been given is that of Hoberman and McKinsey [**9**]. It is hard to classify since it involves a single variable and a variable function. It may be regarded as an infinite set of postulates, all having the same form. In §3 below a single postulate system is given which involves five variables.

As an indication that near maximum condensation is being reached in these sets, we have the result of Diamond and McKinsey [**8**] showing that the use of at least three variables is necessary in describing Boolean algebras.

1. Boolean groups. In this and the next section we consider a set $\mathfrak{S}$ closed under addition. The following notation proves convenient. Let $a + b = (a, b)$, $a + (b, c) = (a, b, c)$ and, in general, $a + (b_1, b_2, \ldots, b_n) = (a, b_1, b_2, \ldots, b_n)$.

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We assume the following identity holds in $\mathfrak{S}$:

P $$(a, b, c, b, c, b, a) = b.$$

It follows at once that

1.1 $$(a, b, a, b, a, b, a) = b.$$

1.2 $$(a + b) + a = b.$$

Proof. By P, notation, and 1.1,

$$\begin{aligned} b &= (a + b, b, a, b, a, b, a + b) \\ &= (a + b) + (b, a, b, a, b, a, b) \\ &= (a + b) + a. \end{aligned}$$

1.3 $$a + (b + a) = b.$$

Proof. By applying 1.2 twice

$$a + (b + a) = [(b + a) + b] + (b + a) = b.$$

1.4 $$c + (b + a) = (a + c) + b.$$

Proof. Starting with P and using 1.2,

$$\begin{aligned} (b, c, b, c, b, a) &= b + a, \\ (c, b, c, b, a) &= (b + a) + b = a, \\ (b, c, b, a) &= a + c, \end{aligned}$$

and $$(c, b, a) = (a + c) + b.$$

From 1.4, 1.2, and 1.3, we have

1.5 $$b = b + (a + a) = (a + a) + b.$$

1.6 $$a + a = b + b.$$

Proof. By 1.2 and 1.5, both sums equal $[b + (a + a)] + b$.

Denoting $a + a$ by O, we have $a + O = O + a = a$. Setting $c = O$ in 1.4, we obtain

1.7 $$a + b = b + a.$$

This, with 1.4, implies

1.8 $$a + (b + c) = (a + b) + c.$$

The proof that $\mathfrak{S}$ is a Boolean group is now complete.

2. Boolean rings. We now assume $\mathfrak{S}$ is closed under multiplication and satisfies, in addition to P, the identity Q given below. The symbol I found in postulate Q represents a fixed element of $\mathfrak{S}$.

Q $$(a,\ (cc)a,\ a(b+c),\ b(cd)) = (ba,\ (II)a,\ d(b+b),\ c(bd)).$$

We use freely the results of the previous section. In particular we note that equal terms may be cancelled if they occur on the same side or opposite sides of an equation.

If $a = d$ and $b = c$ we have from Q that

2.1 $$a + (bb)a = ba + (II)a.$$

Setting $b = I$ in 2.1 we obtain

2.2 $$a = Ia.$$

Hence $(II)a = Ia = a$ and from 2.1 we have

2.3 $$(bb)a = ba.$$

The previous results may be used to give Q the form

2.4 $$(ca,\ a(b+c),\ b(cd)) = (ba,\ dO,\ c(bd)).$$

Setting $a = b = I$ in 2.4 we have

2.5 $$cI + c = dO.$$

Setting $d = I$, we obtain

2.6 $$c = cI.$$

Equations 2.5 and 2.6 imply that

2.7 $$dO = O.$$

Equations 2.3 and 2.6 imply that

2.8 $$bb = b$$

Equation 2.7 may be used to give 2.4 the form

2.9 $$ca + a(b+c) + b(cd) = ba + c(bd).$$

Setting $d = O$, we have

2.10 $$a(b+c) = ba + ca.$$

This and 2.9 imply

2.11 $$b(cd) = c(bd).$$

Setting $d = I$ in 2.11 we find

2.12 $$bc = cb.$$

The last two identities show that multiplication is associative and commutative. Idempotence, the distributive law, and the role of I as a unit are given in 2.8, 2.10, and 2.2. Thus [4, p. 154], $\mathfrak{S}$ is a Boolean ring with unit or a Boolean algebra.

3. A single postulate. We consider a set $\mathfrak{S}$ closed under an operation denoted by a vertical bar. It is convenient to introduce primes to denote

"squares." Thus $a' = a|a$, $a'' = (a')'$ and so on. It is assumed $\mathfrak{S}$ satisfies the following postulate:

R $\qquad (x|(y'|y))'' = (a|(b'|c))''$ implies that $x = (b|a)|(c'|a)$.

If we set $a = x$, $b = c = y$, we obtain

3.1 $$x = (y|x)|(y'|x).$$

Setting $y = x$ we have

3.2 $$x = x'|(x'|x).$$

3.3 $$(a|(b'|c))' = (b|a)|(c'|a)$$

Proof. Set $x = a \mid (b' \mid c)$. By 3.2, $(x' \mid (x' \mid x))'' = (a \mid (b' \mid c))''$. By R, $x' = (b|a)|(c'|a)$.

From 3.1 and 3.3 we have

3.4 $$x = (x|(y'|y))'.$$

3.5 $$x' = y' \text{ implies } x = y.$$

Proof. If $x' = y'$ we have from 3.4 that $(x|(x'|x))'' = (y|(y'|y))''$. By R and 3.2, $x = (y|y)|(y'|y) = y$.

3.6 $$x|(y'|y) = x|(z'|z)$$

Proof. By 3.4, since both terms equal x, $(x|(y'|y))' = (x|(z'|z))'$. We now apply 3.5.

3.7 $$(x|x'')' = x'|(x''|x).$$

Proof. Since $x'' = x'|x'$ this is a consequence of 3.3.

3.8 $$x|x'' = x'.$$

Proof. By 3.7 and 3.3, $(x|x'')'' = (x'|(x''|x))' = (x'|x')|(x'|x') = x'''$. We now apply 3.5.

3.9 $$x''|x = x'$$

Proof. By 3.3 and 3.7, $(x''|x)'' = ((x''|x)|(x''|x))' = (x'|(x''|x))' = (x|x'')''$. We now apply 3.5 and 3.8.

3.10 $$x|x' = x'|x.$$

Proof. By 3.9 and 3.3, $(x|x')' = (x|(x''|x))' = (x'|x)'$. We now apply 3.5.

3.11 $$x'' = x.$$

Proof. By 3.4, 3.3, 3.8, 3.10, 3.6, and 3.2, $x'' = (x''|(x'|x))' = (x|x'')|(x'|x'') = x'|(x'|x'') = x'|(x''|x') = x'|(x'|x) = x$.

3.12 $$x|y = y|x.$$

Proof. $x|y = (x|(y'|y'))'' = ((y|x)|(y''|x))' = (y|x)'' = y|x.$

Identities 3.11, 3.4, and 3.3 are clearly equivalent to Sheffer's three postulates and these together with 3.12 to Bernstein's two postulates [**1**]. Thus $\mathfrak{S}$ is a Boolean algebra under the operations defined by $x + y = (x|y)'$ and $xy = x'|y'$. It is not unlikely that a variation of R would give a single Boolean algebra postulate expressed in terms of complements and either meets or joins.

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A CHARACTERIZATION OF IMPLICATIVE BOOLEAN RINGS

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In the theory of probability, the conditional can be treated by an operation analogous to division. Many properties of the conditional can best be studied by means of the corresponding multiplication (called the cross-product). An implicative Boolean ring is defined [**2**] in terms of a cross-product and the usual Boolean operations. The cross-product is the only device yet known in which the events corresponding to conditional probabilities are themselves elements of the Boolean ring. The fact that such advice was not introduced by Boole is probably the reason why Boolean algebra has been very little used in the theory of probability, although probability was one of the principal applications which Boole had in mind.

When one introduces a cross-product into the two element Boolean algebra, no additional elements are obtained. However the closure under cross-product of any other finite Boolean algebra is infinite. The same is, of course, true for the usual number systems, namely, the set $\{0, 1\}$ is closed under multiplication but the closure of the set obtained by including any additional positive element is infinite. The introduction of probabilities into a Boolean algebra maps the algebra into the reals between 0 and 1. In the case of an implicative Boolean ring, this mapping is a homomorphism in which cross-product corresponds to multiplication of the reals.

It is shown in this paper that implicative Boolean rings can be characterized in terms of familiar ring concepts only. More specifically, a Boolean ring B can contain a cross-product if and only if it is isomorphic to its quotient rings modulo the non-unit principal ideals. The isomorphisms enable one to set up a semigroup of transformations (not necessarily unique) of B into B. These are one-parameter transformations where the parameter is a non-zero element of the Boolean ring. The product of the transformations defines the cross-product of the parameters. The inverse of one of these transformations, when defined, is an implication which is neither strict nor material. The implication can be extended to elements for which the inverse does not exist and can be given a logical interpretation.

By definition a Boolean algebra $(B, \vee, \cdot, \sim)$, or equivalently a Boolean ring with unit $(B, +, \cdot)$ where $+$ denotes symmetric difference, is called *implicative* if there exists a binary operation, $\times$, such that the following postulates hold:

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P0 B is closed under $\times$.
P1 $a \times (b \times c) = (a \times b) \times c$.
P2 $a \times (b + c) = a \times b + a \times c$.
P3 $a \times (b \cdot c) = (a \times b) \cdot (a \times c)$.
P4 $x \neq 0$ and $x \times y = x \times z$ imply $y = z$.
P5 $x \times 1 = x$.
P6 If $x, y \in B$ and $y \neq 0$, then there is an element z such that $x \cdot y = y \times z$. We use the notation $z = x \subset y$ for this element.

Let $x < y$ denote the condition $x \cdot y = x$; let (a) denote the principal ideal generated by a. Then (a) is the set of all $x \in B$ such that $x < a$, and $x \equiv y$ mod (a) if and only if $x \cdot \sim a = y \cdot \sim a$.

THEOREM. *A necessary and sufficient condition for a Boolean ring B to be implicative is that for all $a \in B$, $a \neq 1$, the Boolean rings B and $B/(a)$ are isomorphic.*

Equivalently we may consider $B \cong B/(\sim a)$ for all $a \neq 0$.

Proof of necessity. Let $C = \{x \subset a, x \in B\}$. Then C is a Boolean algebra and $B/(\sim a) \cong C$. For $x \subset a = y \subset a$ is equivalent to each of the following statements: $(x + y) \subset a = 0$; $(x + y) \cdot a = 0$; and $x \equiv y$ mod $(\sim a)$ and thus $x \subset a$ corresponds to that coset of $B/(\sim a)$ of which x is a representative. But for any $y \in B$, $y = (a \times y) \subset a \in C$. Hence $C = B \cong B/(\sim a)$.

Proof of sufficiency. Since $B \cong B/(\sim a)$ for any $a \neq 0$, there exists by hypothesis a one-to-one correspondence between the elements x of B and the cosets which are the elements of $B/(\sim a)$. Let $C_x^{(a)}$ denote the coset corresponding to x. For any element $w \in C_x^{(a)}$, the element $y = w \cdot a$ of $C_x^{(a)}$ is uniquely determined since $v \cdot a = w \cdot a$ if $v, w \in C_x^{(a)}$. This procedure establishes a one-to-one correspondence between the elements x of B and those elements y of B such that $y \cdot a = y$. For, any two different elements y_1, y_2 contained in a must lie in different cosets mod $(\sim a)$ and thus must correspond to different elements of B. The following scheme indicates this one-to-one correspondence, which we denote by T_a, so that $T_a(x) = y$:

$$\begin{array}{l} B \leftrightarrow B/(\sim a) \\ x \leftrightarrow C_x^{(a)} \\ \updownarrow \\ y = w \cdot a \text{ where } w \in C_x^{(a)}. \end{array}$$

The following conditions C0 to C6 will be shown later to correspond to the postulates for implicative Boolean algebra. These conditions are restricted for the present to the single element $a \neq 0$.

C0 $T_a(x) \in B$ for all $x \in B$.

It is more convenient to discuss C1 after the other conditions have been considered.

It follows at once from the definitions of $+$ and $\cdot$ on the elements of $B/(\sim a)$ that

C2 $T_a(x_1 + x_2) = T_a(x_1) + T_a(x_2)$,

C3 $T_a(x_1 \cdot x_2) = T_a(x_1) \cdot T_a(x_2)$.

Since it is shown that T_a is one-to-one, we have

C4 If $a \neq 0$, then $T_a(x_1) = T_a(x_2)$ implies $x_1 = x_2$.

Clearly $T_a(x) = a$ if and only if $x = 1$. For if $x = 1$, the coset $C_1^{(a)}$ in $B/(\sim a)$ corresponding to x must be the unit element of this quotient ring and therefore must consist of all $z \in B$ such that $z \cdot a = a$. The converse follows since T_a is a one-to-one transformation. Thus we have

C5 $T_a(1) = a$.

C6 The transformation $T_a(z) = w$ has an inverse $z = T_a^{-1}(w)$ if $w < a$.

This is assured by the fact that each element of B contained in a is the image under T_a of a unique element z.

We now consider the formulation of condition C1 corresponding to postulate P1. By way of preparation, we shall show how to select a semigroup of the above transformations in which multiplication is consistent.

If $a, b \neq 0$, let $T_a(b) = c$. Then $c \neq 0$ and $c < a$. The transformation T_c which is defined by the given system of isomorphisms is not necessarily the same transformation as $T_a T_b$, where T_a and T_b are defined by the given isomorphisms. In order that the set of all transformations T_a, $a \neq 0$, be consistent, we shall define a transformation T_c havingthe property that $T_c(x) = T_a T_b(x)$.

Since $T_a T_b(1) = T_a(b) = c$, it follows that $T_a T_b(x) < c$, $x \in B$. For the isomorphisms T_a and T_b preserve order, and $x < 1$. If $y < c$ is given, we wish to solve $T_a T_b(x) = y$ for x. To do this we first solve $T_a(u) = y$ for u. This may be done by C6 because $y < a$. We then solve $T_a(x) = u$ for x. In order to be able to do this by C6, it must be shown that $u \cdot b = u$. But this follows from $T_a(u \cdot b) = T_a(u) \cdot T_a(b) = y \cdot c = y = T_a(u)$ and the fact that T_a is one-to-one. Hence the transformation $T_a T_b$ is one-to-one on the set of all elements $x < c$.

Finally, if $T_a T_b(x) = y$ we let $C_x^{(c)} = \{w \in B,\ w \cdot c = y\}$. Then $C_x^{(c)}$ is one of the elements of the quotient algebra $B/(\sim c)$. It is easy to verify that the set of all *these* cosets forms a Boolean algebra isomorphic to B, i.e., that the map $x \to C_x^{(c)}$ is an isomorphism. If we define T_c using *this* system of cosets, then the transformation T_c is consistent with T_a and T_b, whereas the transformation defined by the given isomorphism might be inconsistent with T_a and T_b.

Assuming for the moment that all of the above mentioned inconsistencies have been removed, then to each non-zero element $a \in B$ there corresponds a transformation T_a. The set of all such transformations is a one-parameter semigroup in which the parameter is an element of B. The product of the trans-

formations defines the cross-product of the parameters. More specifically, let $a \times b$ be that element c of B such that $T_a T_b(x) = T_c(x)$. Then

$$T_a(b) = T_a T_b(1) = T_c(1) = c = a \times b.$$

If the non-zero elements of B are well ordered, then the set of corresponding transformations may be made consistent by the following procedure. If a is the first element of B in the well ordering, let T_a be defined as above by means of the given isomorphism, $B \cong B/(\sim a)$. Let $a^1 = a$, and a^n be defined recursively by $a^n = a^{n-1} \times a$, $n > 2$. Then by the argument of the preceding paragraph, the transformations $T_{a^1}, \ldots, T_{a^n}, \ldots$ may all be consistently defined.

Now suppose that the transformations T_x have been defined for all x preceding a given element b in the well-ordered series. Then all finite cross-products of such elements have been defined. If b is one of these cross-products, then T_b is already defined. Otherwise let T_b be defined by the given system of isomorphisms. We have thus obtained a consistent family of transformations T_x for all non-zero $x \in B$. Finally $T_0(z) = 0$ defines T_x for $x = 0$.

Since the consistent family of transformations is a semigroup, we have the condition

C1 $T_a(T_b T_c) = (T_a T_b) T_c$.

Therefore
$$\begin{aligned}[T_a(T_b T_c)]\,(1) &= [(T_a T_b)\,T_c]\,(1),\\ T_a T_{b\times c}(1) &= T_{a\times b} T_c(1),\\ T_a\,(b \times c) &= T_{a\times b}\,(c),\\ a \times (b \times c) &= (a \times b) \times c.\end{aligned}$$

Similarly the remaining conditions C0, C2, C3, C4, C5, C6 are immediately seen from the relation $T_a(b) = a \times b$ to be verifications of the postulates for implicative Boolean algebra. This completes the proof of the characterization theorem.

In C6 we discussed the inverse transformation $z = T_a^{-1}(w)$ defined for $w < a$. We use the notation $z = w \subset a = T_a^{-1}(w)$. Thus z is a function of w and a. This function can be extended so that w can range over the entire ring B. Namely, $z = w \subset a$ is the solution of the equation $w \cdot a = T_a(z)$ for any $w \in B$.

The element $x \subset a$ of an implicative Boolean ring can be interpreted as the sentence "x if a" or the sentence "a implies x." This "if" operation is the conditional in probability theory. The conventional treatment of the conditional is based on ordered pairs of propositions, whereas in our system such an ordered pair is itself a proposition, i.e., an element of the ring. Koopman [**4**] uses this particular implication as a model for conditional probability, but still treats the conditional as an ordered pair of propositions. Material implication "a implies x" is the proposition $\sim a \vee x$. Strict implication can be defined equivalently by either of the equations $\sim a \vee x = 1$ or $x \subset a = 1$. These three implications are all distinct. Our implication $x \subset a$ is the only implication which is appropriate to the theory of probability.

In addition, the implication $x \subset a$ has an interpretation in formal logic. Consider a set of postulates $P_1, P_2, \ldots, P_n$. Let B be the set of all propositions which are statable on the base of these postulates. We extend B to an implicative Boolean ring B^* by the method of [**3**]. Let $P = P_1 \cdot P_2 \cdot \ldots \cdot P_n$. These elements of B which are in the unit coset of $B^*/(\sim P)$ are those which are strictly implied by the postulates. For $x \subset P = 1$ if and only if x is the unit coset of $B^*/(\sim P)$.

It is meaningful in this extended language B^* to consider propositions of the form $x \subset P$ where x is not necessarily an element of the unit coset of $B^*/(\sim P)$. That is, we consider such implications as valid sentences even though they may not be true in the sense of being strict implications. This last property is also true of material implication. The language B^* is thus seen to be a metalanguage containing the original language B. In the probabilistic interpretation, this metalanguage also contains all conditional sentences.

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EXTENSIONS OF LIE ALGEBRAS AND THE THIRD COHOMOLOGY GROUP

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Introduction. Cohomology theories of various algebraic structures have been investigated by several authors. The most noteworthy are due to Hochschild, MacLane and Eckmann, Chevalley and Eilenberg, who developed the theory of cohomology groups of associative algebras, abstract groups, and Lie algebras respectively. In this paper we are concerned primarily with a characterization of the third cohomology group of a Lie algebra by its extension properties.

In §1 necessary definitions from Chevalley and Eilenberg's theory are given [2]; §2 is concerned with a special type of extension. In §3 we define the invariant coboundary: a mapping of $H^q(L, P)$ into $H^{q+1}(L, Q)$ for any representation modules $\{V, P\}$ and $\{W, Q\}$ of L. In §4 we consider the special extension problem corresponding to the Teichmüller theory for simple (associative) algebras [4].

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1. Definition of the cohomology groups. Let L be a Lie algebra over a field F, and P a representation of L by means of linear endomorphisms of a vector space V of finite dimension over F. A q-linear alternating mapping of L into V will be called a q-dimensional V-cochain (or shorter: a q-V-cochain). The q-V-cochains form a space $C^q(L, V)$. By definition, $C^0(L, V) = V$. We define a linear mapping $f \to \delta f$ of $C^q(L, V)$ into $C^{q+1}(L, V)$ by the formula

$$(\delta f)(x_1, \ldots, x_{q+1}) = \sum_{k<l} (-1)^{k+l+1} f([x_k, x_l], x_1, \ldots, \tilde{x}_k, \ldots, \tilde{x}_l, \ldots, x_{q+1}) + \sum_{i=1}^{q+1} (-1)^{i+1} P(x_i) f(x_1, \ldots, \tilde{x}_i, \ldots, x_{q+1}),$$

where the tilde implies omission of the corresponding variable. If $q = 0$ then $f \in V$ and δf is defined by $(\delta f)(x) = P(x)f$. For any $f \in C^q(L, V)$ and all q, $\delta\delta f = 0$. A cochain f is a cocycle provided $\delta f = 0$. The cocycles of dimension q form a subspace $Z^q(L, P)$ of $C^q(L, V)$. A cochain $f \in C^q(L, V)$ is a coboundary if it is of the form δg for some $g \in C^{q-1}(L, V)$. The coboundaries of dimension

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q form a subspace $B^q(L, P)$ of $Z^q(L, P)$. By definition $B^0(L, P) = \{0\}$. The factor space

$$H^q(L, P) = Z^q(L, P)/B^q(L, P)$$

is called the qth cohomology group of L by P.

2. The extension $U = (L, V, W, \beta)$. Let S be an arbitrary set of elements. By an *S-module* on a field F we mean a pair $\{V, P\}$ formed by a vector space V of finite dimension over F and a mapping P which assigns to every element $x \in S$ a linear endomorphism $P(x)$ of V. In particular, let S be the set of elements of a Lie algebra L. An S-module $\{V, P\}$ is called a *representation module* of L if the following condition is satisfied:

$$P([x, y]) = P(y)\, P(x) - P(x)\, P(y)$$

for any elements x, y of L. In this case the mapping P is called a *representation* of L.

The group $H^2(L, P)$ was related by Chevalley and Eilenberg to the extension L by P as follows: We define an *extension* $L^+ = (L, V)$ of L by P to be a Lie algebra with the following properties:

(i) V is an ideal in L^+,

(ii) $[V, V] = 0$, that is, V is an abelian ideal,

(iii) $L^+/V \cong L$,

(iv) The *linear representatives* $\rho_x = (\rho(x)) \in L^+$ corresponding to $x \in L$ by the isomorphism (iii) satisfy[1] $P_x v = [v, \rho(x)]$.

The structure of L^+ is completely determined by

$$[\rho_x, \rho_y] = \rho_{[x,y]} + g(x, y), \qquad x, y \in L, \quad g(x, y) \in V,$$

where g satisfies the condition corresponding to

$$[[\rho_x, \rho_y], \rho_z] + [[\rho_y, \rho_z], \rho_x] + [[\rho_z, \rho_x], \rho_y] = 0,$$

that is,

$$g([x, y], z) + g([y, z], x) + g([z, x], y) + P_z g(x, y) + P_x g(y, z) + P_y g(z, x) = 0.$$

Hence g is a 2-P-cocycle. Conversely, for any given $g \in Z^2(L, P)$ there exists an extension L^+ with this g. We denote this extension by $L^+ = (L, V, g)$. If we choose another system of representatives

$$\rho_x^+ = \rho_x + h(x) \qquad (h(x) \in V),$$

the corresponding g^+ is given by

$$g^+(x, y) = g(x, y) + \{P_y h(x) - P_x h(y) - h([x, y])\},$$

namely $g^+ \equiv g(\text{mod } B^2(L, P))$. Hence cohomologous g's generate isomorphic extensions. The extension L^+ is said to *split* if there is a subalgebra L' of L^+

[1] Let ϕ be a homomorphism of L^+ onto L. The representatives ρ_x (ρ is a linear function of x) are any fixed set of elements of L^+ satisfying $\phi\rho_x = x$ and $\rho_0 = 0$. Furthermore, V is the kernel of the homomorphism ϕ.

such that ϕ maps L' isomorphically onto L. Hence the vanishing of $H^2(L, P)$ for all P implies the splitting of all extensions $L^+ = (L, V)$.

Let the pair $\{U, R\}$ be a representation module of L with an L-invariant submodule W (with the operation of L on W denoted by Q). On the factor space U/W one has then an induced operation by L; if this is isomorphic to a module $\{V, P\}$, we call U an *extension of V by W with respect to L*.

Denote the elements of L by $x, y, \ldots$ and those of V by $v_1, v_2, \ldots$. For each element $v \in V$ we take a representative $\mu_v \in U$ from the residue class corresponding to $v \in V$ by the isomorphism $U/W \cong V$ such that μ_v depends linearly on v. Hence

$$U = (W + O) \cup (W + \mu_{v_1}) \cup (W + \mu_{v_2}) \cup \ldots$$

where O is the zero representative and

$$R_x\mu_v = \mu_{P_xv} + \beta(x, v), \qquad \beta(x, v) \in W. \tag{2.1}$$

It follows from equation 2.1 that β is a bilinear function of $x \in L$ and $v \in V$. Now, since R is a representation on U,

$$R_y R_x \mu_v - R_x R_y \mu_v = R_{[x,y]}\mu_v$$

for all $x, y \in L$ and $v \in V$. Hence

$$\beta(x, P_yv) - \beta(y, P_xv) + \beta([x, y], v) + Q_x\beta(y, v) - Q_y\beta(x, v) = 0 \tag{2.2}$$

for all $x, y \in L$ and $v \in V$. If we choose another set of linear representatives

$$\mu_v^+ = \mu_v + K_v, \qquad v \in V, \quad K_v \in W,$$

we have

$$\beta^+(x, v) = \beta(x, v) + \{Q_x K_v - K_{P_xv}\}. \tag{2.3}$$

We call β satisfying 2.2 a factor system and denote it by $\{\beta\}$. Two factor systems $\{\beta\}$ and $\{\beta^+\}$ satisfying the relation 2.3 are said to be *associated*. The structure of an extension U is completely determined by the factor system $\{\beta\}$. Hence we write $U = (L, V, W, \beta)$. Conversely, for any factor system $\{\beta\}$ there exists an extension $U = (L, V, W, \beta)$ satisfying 2.1 Two extensions $U_i = (L, V, W, \beta_i)$ $(i = 1, 2)$ are isomorphic (as L-modules, each element of $W < U_i$ $(i = 1, 2)$ corresponding to itself) if and only if $\{\beta_1\}$ and $\{\beta_2\}$ are associated. In this case we identify U_1 with U_2.

We define $\{\beta_1 + \beta_2\} = \{\beta_1\} + \{\beta_2\}$. Then all the factor systems form a module $\Phi(V, W)$. In a splitting factor system there is a set of representatives μ_v such that $\beta(x, v) = 0$. Then

$$\beta^+(x, v) = Q_xK_v - K_{P_xv}.$$

The splitting factor systems form a subspace $\Sigma(V, W)$ of $\Phi(V, W)$. Hence we have

THEOREM 2.4. *The elements of $\Phi(V, W)/\Sigma(V, W)$ correspond in a one-to-one manner with the extensions U of V by W with respect to L.*

3. The invariant coboundary. This section is merely an adaptation of the well-known relative cohomology sequence for coefficients to Lie algebras. Given the extension $U = (L, V, W, \beta)$, it is known that there exists a relative cohomology sequence[2] of homomorphisms,

3.1
$$\ldots \to H^q(L, V) \xrightarrow{\Lambda} H^{q+1}(L, W) \xrightarrow{\mathrm{M}} H^{q+1}(L, U) \xrightarrow{\mathrm{N}} H^{q+1}(L, V) \to H^{q+2}(L, W) \to \ldots,$$

and that this sequence is *exact*. In this sequence the mapping M is the obvious one: regard a cochain with values in W as if it has values in the larger module U; N is also obvious: take a cochain with values in U and reduce the values modulo W to obtain one with values in V. The mapping Λ is usually called the *invariant coboundary* and is described normally as follows: Let $\psi: U \to V$ be the given homomorphism of U upon its quotient $V \cong U/W$, and let g be any cocycle in $C^q(L, V)$. Pick representatives $\bar{g}(x_1, \ldots, x_q)$ at random so that $\bar{g}$ is multilinear and

$$\psi\bar{g}(x_1, \ldots, x_q) = g(x_1, \ldots, x_q).$$

Then $f = \delta\bar{g}$ actually has values in W and the mapping Λ is the one obtained by sending the cohomology class of g in $H^q(L, V)$ into that of f in $H^{q+1}(L, W)$.

We define a linear mapping $F = F_\beta$ of $C^q(L, V)$ into $C^{q+1}(L, W)$ $(q > 0)$ as follows:

$$F_\beta(g) = f \in C^{q+1}(L, W), \qquad g \in C^q(L, V),$$

where

$$f(x_1, \ldots, x_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1}\beta(x_i, g(x_1, \ldots, \hat{x}_i, \ldots, x_{q+1})), \quad x_1, \ldots, x_{q+1} \in L.$$

A minor computation shows that the mapping Λ is essentially the same as the mapping F_β when applied to cocycles g. When applied to cochains g, the two maps differ by

$$\mu(\delta g)(x_1, \ldots, x_{q+1}).$$

The advantage of the invariant coboundary is that it avoids some of the long computations necessary when employing the mapping F_β. For example, it is not necessary to prove that the mappings F_β and δ commute. Also the proof that the sequence 3.1 is exact in the usual sense is entirely straightforward. The map Λ is defined for the extension (an easy argument shows that the choice of representatives does not matter)[3] and not from the factor sets β or β^+.

If the factor system $\{\beta\}$ splits, then $\beta = 0$, and so also $F_\beta = 0$. Hence:

THEOREM 3.2 *If the factor system $\{\beta\}$ splits, then Λ maps $H^q(L, V)$ into the zero cohomology class of $H^{q+1}(L, W)$ $(q > 1)$.*

4. The group $H^3(L, W)$. An interpretation of the third cohomology group in relation and analogous to the Teichmüller theory of factor systems of

[2]We use U, V, W here instead of R, P and Q respectively. The cohomology groups have the same meaning as before.

[3]If the factor systems $\{\beta\}$ and $\{\beta^+\}$ are associated, then F_β and F_{β^+} induce the same mapping of $H^q(L, V)$ into $H^{q+1}(L, W)$.

higher degree is now given. Let $L^+ = (L, V, g)$ be an extension of L with factor set g and $U = (L, V, W, \beta)$ an extension of V by W with respect to L. The extension U implies the existence of representation modules $\{U, R\}$ and $\{V, P\}$ of L satisfying

4.1 $$R_x\mu_v = \mu_{P_x v} + \beta(x, v), \qquad x \in L,\ v \in V,\ \mu_v \in U,\ \beta(x, v) \in W$$

(cf. equation 2.1) where the elements μ_v are the representatives corresponding to the isomorphism $U/W \cong V$. We consider the following problem.

To construct[4] *an extension $L^{++} = (L, U)$ of L by R satisfying $L^{++}/W \cong L^+$.* Suppose that we have such an extension. We then have the following lattice:

$$\begin{array}{ccccc}
 & \xleftarrow{\ \tau\ } & & \xleftarrow{\ \rho\ } & \\
L^{++} & \longrightarrow & L^{+} & \longrightarrow & L \\
| & \xleftarrow{\ \mu\ } & | & & | \\
U & \longrightarrow & V & \longrightarrow & 0 \\
| & & | & & \\
W & \longrightarrow & 0 & &
\end{array}$$

From $L^{++}/W \cong L^+$ choose linear representatives

$$\tau_{x^+} \in L^{++} \qquad (x^+ \in L^+)$$

such that

$$\tau_{x^+} \equiv \mu_{x^+}$$

on V. Hence

$$L^{++} = (W + \tau_{x^+}) \cup (W + \tau_{y^+}) \cup \ldots, \qquad x^+, y^+, \ldots \in L^+;\ \ \tau_{x^+}, \tau_{y^+}, \ldots \in L^{++}.$$

Then

4.2 $$[\tau_{x^+}, \tau_{y^+}] = \tau_{[x^+, y^+]} + l(x^+, y^+), \qquad l(x^+, y^+) \in W.$$

In particular,

$$\begin{aligned}
[\tau_{\rho_x}, \tau_{\rho_y}] &= \tau_{[\rho_x, \rho_y]} + l(\rho_x, \rho_y) \\
&= \tau_{\rho[x,y]+g(x,y)} + l(\rho_x, \rho_y) \qquad (4.3)\\
&= \tau_{\rho[x,y]} + \mu_{g(x,y)} + l(\rho_x, \rho_y), \qquad l(\rho_x, \rho_y) \in W
\end{aligned}$$

where the representatives $\rho_x \in L^+$ are selected in such a way that

$$[\rho_x, \rho_y] = \rho_{[x,y]} + g(x, y).$$

Now, from the isomorphism $L^{++}/U \cong L$ choose linear representatives $\sigma_x \in L^{++}$ so that $\sigma_x = \tau(\rho_x)$. Then

$$L^{++} = (U + \sigma_x) \cup (U + \sigma_y) \cup \ldots, \qquad x, y, \ldots \in L;\ \ \sigma_x, \sigma_y, \ldots \in L^{++}.$$

Therefore we have the following multiplication of representatives

4.4 $$[\sigma_x, \sigma_y] = \sigma_{[x,y]} + \mu_{h(x,y)} + \alpha(x, y), \qquad \alpha(x, y) \in W,\ \ h \in C^2(L, V).$$

[4] L^{++}/W may then be regarded as an extension of L.

Comparing equations 4.3 and 4.4 we observe that $\mu_{h(x,y)} = \mu_{g(x,y)}$, so that

4.4′ $$[\sigma_x, \sigma_y] = \sigma_{[x,y]} + \mu_{g(x,y)} + \alpha(x, y), \qquad \alpha(x, y) \in W.$$

Since $g \in Z^2(L, P)$ it is easy to see that the function α is a 2-W-cochain.

The extension L^{++} of L by R implies that $R_x u = [u, \sigma_x]$ $(u \in U)$, and so, by 4.1,

4.5 $$[\mu_v, \sigma_x] = \mu_{P_x v} + \beta(x, v), \qquad \beta(x, v) \in W.$$

From equations 4.4 and 4.5,

4.6 $$\begin{aligned}[][\sigma_x, [\sigma_y, \sigma_z]] &= [\sigma_x, \sigma_{[y,z]} + \mu_{g(y,z)} + \alpha(y, z)] \\ &= \sigma_{[x,[y,z]]} + \mu_{g(x,[y,z])} + \alpha(x, [y\ z]) \\ &\quad - \mu_{P_x(g(y,z))} - \beta(x, g(y, z)) - Q_x\alpha(y, z)\end{aligned}$$

since $[w, \sigma_x] = R_x w \equiv Q_x w$ $(w \in W)$. The Jacobi identity for the representatives σ_x yields symbolically

4.7 $$\Lambda(g) + \delta\alpha = 0.$$

Denote by $k(\in Z^2(L, U))$ the factor set belonging to L^{++}. Then

$$k(x, y) = \mu_{g(x,y)} + \alpha(x, y) = \bar{g}(x, y) + \alpha(x, y),$$

and so $\delta\alpha = -\delta\bar{g}$. Hence $\Lambda(g) = \delta\bar{g}$, and in addition $k(x, y) \equiv \bar{g}(x, y) \bmod W$.

Conversely, if we have $\alpha(x, y) \in W$ satisfying equation 4.7 then we can construct the extension L^{++} as follows: To each $x \in L$ assign a symbol σ_x. The algebra L^{++} is to consist of all the elements of all the cosets $U + \sigma_x$. Multiplication of two σ_x's will be defined by 4.4′ and the multiplication of a σ_x and a μ_v by the equation 4.5. Multiplication of σ_x and w is defined by $[w, \sigma_x] = Q_x w$. Since Q_x is a linear endomorphism, W is an ideal in L^{++}. Further, for an arbitrary representative $\mu_v \in U$ and $w \in W$, $[\mu_v, w] = 0$ since U is abelian. There remains the verification of the Jacobi identity for the σ_x and this is equivalent to 4.7. We must also verify the Jacobi identities for mixed multiplications of σ_x's and μ_v's:

LEMMA 4.8.

(i) $$[\sigma_x, [\mu_{v_1}, \mu_{v_2}]] + [\mu_{v_1}, [\mu_{v_2}, \sigma_x]] + [\mu_{v_2}, [\sigma_x, \mu_{v_1}]] = 0,$$

(ii) $$[\sigma_x, [\mu_v, \sigma_y]] + [\mu_v, [\sigma_y, \sigma_x]] + [\sigma_y, [\sigma_x, \mu_v]] = 0.$$

The proof of (i) is obvious since U is abelian.

Proof of (ii). The expression on the left is equal to

$$\begin{aligned}&[\sigma_x, \mu_{P_y v} + \beta(y, v)] + [\mu_v, \sigma_{[y,x]} + \mu_{g(y,x)} + \alpha(y, x)] + [\sigma_y, -\mu_{P_x v} - \beta(x, v)] \\ &= -\mu_{P_x(P_y v)} - \beta(x, P_y v) - Q_x\beta(y, v) + \mu_{P_{[y,x]}v} + \beta([y, x], v) + \mu_{P_y(P_x v)} \\ &\qquad\qquad + \beta(y, P_x v) + Q_y\beta(x, v) \\ &= -\beta(x, P_y v) + \beta(y, P_x v) + \beta([y, x], v) - Q_x\beta(y, v) + Q_y\beta(x, v) = 0,\end{aligned}$$

by the relation 2.2. Hence we have

THEOREM 4.9. *Let L be a Lie algebra over a field F and $\{U, R\}$ a representation module of L where $U = (L, V, W, \beta)$ is an extension of V by W with respect to L. Then for a given extension $L^+ = (L, V, g)$ a necessary and sufficient condition for the existence of another extension $L^{++} = (L, U)$ of L by R such that $L^{++}/W \cong L^+$ is that the 3-Q-cocycle $\Lambda(g)$ is a coboundary.*

COROLLARY 4.91. *If $H^3(L, Q) = \{0\}$ then there is always such an extension.*

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FOCAL SERIES IN FINITE GROUPS

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If there is given a subgroup S of a (finite) group G, we may ask what information is to be obtained about the structure of G from a knowledge of the location of S in G. Thus, for example, famed theorems of Frobenius and Burnside give criteria for the existence of a normal subgroup N of G such that $G = NS$ and $1 = N \cap S$, and hence in particular for the non-simplicity of G. To aid in locating S in G, and to facilitate exploitation of the transfer, we single out a descending chain of normal subgroups of S. Namely, we introduce the *focal series* of S in G by means of the recursive formulae

${}_0S = S$, ${}_{i+1}S =$ *the subgroup of* G *which is generated by all commutators* $c = [s, g]$, *with* c *and* s *in* ${}_iS$, g *in* G.

For π a set of primes, let us denote by $P(\pi)$ the subgroup of G which is generated by all those elements of G whose orders have no prime divisors in π. A central theorem of our discussion is the following: if π contains all the prime divisors of the index $[S: {}_iS]$ for some i, then every prime which divides $[P(\pi) \cap S: P(\pi) \cap {}_iS]$ also divides $[P(\pi): P(\pi) \cap S]$. From this result we deduce in particular a generalization of Burnside's theorem: if ${}_iS = 1$ for some i, in which case we call S *hyperfocal*, and if S has order prime to its index in G, then there exists a normal subgroup N of G such that $G = NS$ and $1 = N \cap S$. Thus a group is nilpotent if and only if each of its Sylow subgroups is hyperfocal.

The only proof of the theorem of Frobenius in the literature involves group characters. In an attempt to give a purely group theoretic proof, Grün has established a generalization of a special case of this theorem. Our methods lead to a somewhat sharpened form of the theorem of Grün.

Notations. Throughout we shall write *group* for *finite group*. If G is a group,

$Z(G) =$ the centre of G.

${}^iG =$ the ith term of the lower central series of G.

If S is a subgroup of G,

$N(S) =$ the normalizer of S in G.

$C(S) =$ the centralizer of S in G.

When it becomes necessary to emphasize the role of G, we write

$N(S) = N(S \text{ in } G)$, $C(S) = C(S \text{ in } G)$.

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For a subset X and an element g of G,
X^g = the totality of elements $g^{-1}xg$, for x in X.
If A and B are two groups,
$A \otimes B$ = the direct product of A and B.
For two sets X and Y,
$X \subseteq Y$ means that X is part of Y.
$X \subset Y$ means that X is a proper part of Y.
$X \cap Y =$ the totality of elements contained in both X and Y.

1. The subgroups $P(\pi)$ and $G^t(\pi)$. This section is devoted to preliminaries. In particular we define certain subgroups which play an important role in our subsequent discussion. We begin by proving

LEMMA 1.1. *If N is a normal subgroup and S a subgroup of the group G such that $G = NS$, then*
(a) *${}^tS \subseteq N$ if and only if ${}^tG \subseteq N$.*
(b) *${}^tS \subseteq N \cap S \subseteq {}^tG$ if and only if ${}^tG \cap S = N \cap S$.*
(c) *If H is a normal subgroup of S, then NH is a normal subgroup of G.*

Proof. Since $G = NS$, G/N is isomorphic with $S/[N \cap S]$. If ${}^tS \subseteq N$, then ${}^tS \subseteq N \cap S$, and hence it follows that ${}^tG \subseteq N$. On the other hand, since ${}^tS \subseteq {}^tG$, the converse is clear.

By (a), if ${}^tS \subseteq N \cap S$, then ${}^tG \subseteq N$. Hence, if in addition $N \cap S \subseteq {}^tG$, we have ${}^tG \cap S \subseteq N \cap S \subseteq {}^tG \cap S$, so that ${}^tG \cap S = N \cap S$. If conversely we have ${}^tG \cap S = N \cap S$, then ${}^tS \subseteq {}^tG \cap S = N \cap S \subseteq {}^tG$. Hence (b).

To prove (c) we need only show that $H^x \subseteq NH$ for each x in G. But $G = NS$, hence $x = sn$, with s in S and n in N. Thus, since H is a normal subgroup of S, $H^x = H^{sn} = H^n \subseteq NH$. This completes the proof of the lemma.

Now let π be a set of primes. We call a group (element of a group) a π-group (π-element) if all the prime divisors of its order are in π. For G a group we denote by $P(\pi)$ the subgroup of G which is generated by all those elements of G whose orders have no prime divisors in π. Clearly $P(\pi)$ is a fully invariant subgroup of G. Furthermore,

1.2 *$P(\pi)$ is the intersection of all normal subgroups N of G such that G/N is a π-group.*

To verify (1.2), notice first that $G/P(\pi)$ is a π-group, and hence that $P(\pi)$ contains this intersection. If on the other hand N is a normal subgroup of G such that G/N is a π-group, and if p is a prime not in π such that p^a divides the order of G, then p^a divides the order of N. Hence, since N is a normal subgroup of G, N contains all the p-Sylow subgroups of G. Thus we conclude that $P(\pi) \subseteq N$, which completes the verification. We also note the following simple fact:

1.3 *If S is a subgroup of G, and if π is a set of primes which contains no prime divisor of $[G: S]$, then $G = P(\pi)S$.*

For if p is a prime which is not in π, $P(\pi)$ contains (all) p-Sylow subgroups of G, whereas if p is in π, and p^a divides the order of G, then, since p does not divide $[G: S]$, p^a divides the order of S. This proves 1.3.

Now we introduce an additional definition, namely we set

$G^i(\pi) =$ the intersection of all normal subgroups N of G such that G/N is a π-group, and ${}^iG \subseteq N$.
$\quad = $ the intersection of all normal subgroups N of G such that $[G: N]$ is a power of some prime in π, and G/N has a lower central series of length at most i.

That these are indeed two descriptions of the same subgroup is immediate when we remember that if G/N is a nilpotent π-group, then N is the intersection of all normal subgroups M of G which contain N, and have index in G equal to a power of some prime in π.

The subgroup $G^i(\pi)$ is characteristic in G, $G/G^i(\pi)$ is a π-group, and $G^i \subseteq G^i(\pi)$, i.e. $G/G^i(\pi)$ is nilpotent, with a lower central series of length at most i. By 1.2 we have the useful formula

1.31 $\quad G^i(\pi) = {}^iG\,P(\pi)$.

Since furthermore $G^i(\pi)/{}^iG$ is the π-complement of the nilpotent group $G/{}^iG$,

1.4 *no prime divisor of* $[G^i(\pi): {}^iG]$ *is in* π.

The factor group $G/G^1(\pi)$ is an abelian π-group; $G^1(\pi)$ is called the *π-commutator subgroup* of G. There exists a k such that $G^k(\pi) = G^{k+1}(\pi) =$ the intersection of all normal subgroups N of G such that G/N is a nilpotent π-group; $G/G^k(\pi)$ is the so-called *maximal nilpotent π-factor group* of G.

LEMMA 1.5. *For a set π of primes, and a subgroup S of the group G, the following three conditions are equivalent:*

(i) *π contains all the prime divisors of the index* $[G^i(\pi) \cap S: {}^iG \cap S]$.

(ii) $G^i(\pi) \cap S = {}^iG \cap S$.

(iii) *π contains all the prime divisors of the index* $[S: {}^iG \cap S] = [{}^iGS: {}^iG]$.

Proof. $G^i(\pi)/{}^iG$ contains ${}^iG[G^i(\pi) \cap S]/{}^iG$, and this latter group is isomorphic with $[G^i(\pi) \cap S]/[{}^iG \cap S]$. Hence, since the order of $G^i(\pi)/{}^iG$ has no prime divisors in π, it is clear that (i) implies (ii).

$[S: S \cap G^i(\pi)] = [G^i(\pi)S: G^i(\pi)]$ is a divisor of $[G: G^i(\pi)]$. Hence (ii) implies (iii), since $G/G^i(\pi)$ is a π-group.

Finally, it is clear that (i) is a consequence of (iii).

PROPOSITION 1.6. *For π a set of primes, and S a subgroup of the group G, the following three conditions are equivalent:*

(i) $G = G^i(\pi)S$ *and* ${}^iG \cap S = G^i(\pi) \cap S$.

(ii) *π contains all the prime divisors of* $[{}^iGS: {}^iG]$, *but none of* $[G: {}^iGS]$.

(iii) $G = G^j(\pi)S$ *and* ${}^jG \cap S = G^j(\pi) \cap S$, *for all* $j \leq i$.

Hence there exist a set π of primes which satisfies the equivalent conditions (i), (ii), *and* (iii) *if and only if the indices* $[G: {}^iGS]$ *and* $[{}^iGS: {}^iG]$ *are relatively prime.*

Proof. Assume (i), then by Lemma 1.5, all the prime divisors of $[{}^iGS: {}^iG]$ are in π. And

$$[G: {}^iGS] = [G^i(\pi): G^i(\pi) \cap {}^iGS] = [G^i(\pi): {}^iG(G^i(\pi) \cap S)]$$
$$= [G^i(\pi): {}^iG({}^iG \cap S)] = [G^i(\pi): {}^iG],$$

hence by 1.4, $[G: {}^iGS]$ has no prime divisors in π. Hence (i) implies (ii).

Now asume (ii). By Lemma 1.5, ${}^iG \cap S = G^i(\pi) \cap S$. Furthermore, if p is a prime in π, p does not divide $[G: {}^iGS]$, and hence if p^a divides the order of G, p^a divides the order of ${}^iGS$. If on the other hand q is a prime which is not in π, $P(\pi)$ contains (all) q-Sylow subgroups of G. Hence $G = P(\pi){}^iGS = G^i(\pi)S$, by formula 1.31. Hence (ii) implies (i), and we have proved the equivalence of (i) and (ii).

If $j < i$, $[G: {}^jGS]$ is a divisor of $[G: {}^iGS]$, and $[{}^jGS: {}^jG]$ is a divisor of $[{}^iGS: {}^iG]$. Hence if π satisfies (ii), π contains all the prime divisors of $[{}^jGS: {}^jG]$, but none of $[G: {}^jGS]$. Hence, since (ii) implies (i), (ii) implies (iii). And it is clear that (iii) implies (i). Thus the three conditions are equivalent as stated.

It is clear that the existence of a set π of primes satisfying (ii) implies that $[G: {}^iGS]$ and $[{}^iGS: {}^iG]$ are relatively prime. If on the other hand these indices are relatively prime, then the totality of prime divisors of $[{}^iGS: {}^iG]$ satisfies (ii).

Proposition 1.7. *Let S be a subgroup of the group G, and assume that there exists a set π of primes such that $G = G^i(\pi)S$ and ${}^iG \cap S = G^i(\pi) \cap S$. Then if N is a normal subgroup of G such that $G = NS$ and ${}^jG \cap S = N \cap S$, for some $j < i$, $N = G^j(\pi)$.*

Proof. In view of Proposition 1.6 it will suffice to consider the case $i = j$. Assume then that N is a normal subgroup of G such that $G = NS$ and ${}^iG \cap S = N \cap S$. Since ${}^iS \subseteq {}^iG \cap S = N \cap S$, we have ${}^iG \subseteq N$ by Lemma 1.1(a). By Lemma 1.5, π contains all the prime divisors of $[{}^iGS: {}^iG]$, hence since $[G: N] = [S: N \cap S] = [S: {}^iG \cap S]$, G/N is a π-group. Hence N contains $G^i(\pi)$, and hence $N = G^i(\pi)$, since

$$G/N \simeq S/[{}^iG \cap S] \simeq G/G^i(\pi).$$

In the next section we shall introduce the focal series of S in G, the ith term ${}_iS$ of which will be a normal subgroup of S, such that ${}^iS \subseteq {}_iS \subseteq {}^iG$. In §3 we shall obtain a useful sufficient condition for the validity of the relation $SG = P(\pi)S$ and $P(\pi) \cap S \subseteq {}_iS$. Here we prove

Proposition 1.8. *Let H be a normal subgroup of the subgroup S of G, such that ${}^iS \subseteq H \subseteq {}^iG$. Then for π a set of primes, the following conditions are equivalent:*

(A) *$G = P(\pi)S$ and $P(\pi) \cap S \subseteq H$.*

(B) *$N = P(\pi)H$ is a normal subgroup of G such that $G = NS$ and $H = N \cap S$.*

(C) *$H = {}^tG \cap S = G^t(\pi) \cap S$, ${}^tG \subseteq P(\pi)H$, and $G = G^t(\pi)S$.*
(D) *$H = {}^tG \cap S$, ${}^tG \subseteq P(\pi)H$, and π contains all the prime divisors of $[{}^tGS: {}^tG]$, but none of $[G: {}^tGS]$.*
(E) *$H = {}^tG \cap S$, $G = P(\pi)S$, and π contains all the prime divisors of $[{}^tGS: {}^tG]$.*
(F) *$P(\pi) \cap S \subseteq H$, $G = P(\pi)S$, and no prime divisor of $[G: {}^tGS]$ is in π.*

Proof. Assume (A). Then $N = P(\pi)H$ is a normal subgroup of G by Lemma 1.1 (c). Furthermore, $NS = P(\pi)HS = P(\pi)S = G$, and $N \cap S = P(\pi)H \cap S = [P(\pi) \cap S]H = H$. Hence (B) is a consequence of (A).

Now assume (B). Since ${}^tS \subseteq H = N \cap S \subseteq {}^tG$, and $G = NS$, we have by Lemma 1.1 (a) and (b) that ${}^tG \subseteq N$, and ${}^tG \cap S = N \cap S = H$. Since ${}^tG \subseteq N = P(\pi)H$, $G^t(\pi) = P(\pi){}^tG \subseteq N \subseteq G^t(\pi)$, so that $N = G^t(\pi)$. Now we see that (B) implies (C). The equivalence of (C) and (D) is a consequence of Proposition 1.6.

Now it is easy to prove that (C) implies (A), from which the equivalence of the first four conditions will follow. Assume (C). Then $G^t(\pi) = P(\pi)H$ since ${}^tG \subseteq P(\pi)H$, hence

$$P(\pi)S = P(\pi)HS = G^t(\pi)S = G,$$

and

$$P(\pi) \cap S \subseteq P(\pi)H \cap S = G^t(\pi) \cap S = H.$$

Hence (C) implies (A).

If the equivalent conditions (A), (C), and (D) are satisfied, then it is clear that (E) and (F) are also satisfied. Assume (E), then it follows, using Lemma 1.5 that

$$H = {}^tG \cap S = G^t(\pi) \cap S = P(\pi)\,{}^tG \cap S \supseteq P(\pi) \cap S.$$

Hence (E) implies (A). Finally assume (F). Then $H \subseteq {}^tG \cap S \subseteq P(\pi)H \cap S = H$, hence $H = {}^tG \cap S$. Moreover,

$$[{}^tGS: {}^tG] = [S: S \cap {}^tG] = [S: H] = [P(\pi)S: P(\pi)H],$$

from which it follows that π contains all the prime divisors of $[{}^tGS: {}^tG]$. Hence (F) implies (D). We have now proved the equivalence of the six conditions as stated.

COROLLARY 1.9. *Assuming the hypotheses of proposition* 1.8, *we have*
(a) *$[G: {}^tGS]$ and $[S: H]$ are relatively prime.*
(b) *For a normal subgroup N of G, the following conditions are equivalent:*
(i) *$G = NS$ and $H = N \cap S$.*
(ii) *$N = G^t(\pi) = P(\pi)H$.*
(iii) *$N = G^t(\sigma)$, σ a set of primes which contains all the prime divisors of $[S: H]$, but none of $[G: {}^tGS]$.*

Proof. By Proposition 1.8, π contains all the prime divisors of $[S: H]$, but none of $[G: {}^tGS]$. Hence (a).

If we set $N = P(\pi)H$, then we have by proposition 1.8, that $G = NS$, $H = N \cap S$, and $N = G^i(\pi)$. Now (b) is an easy consequence of Propositions 1.6 and 1.7.

2. The focal series. If X, Y, and Z are subgroups of a group G, we shall denote by $[X, Y; Z]$ the subgroup of G which is generated by all the commutators $[x, y] = x^{-1}y^{-1}xy$, with x in X and y in Y, which are in Z. In case $Z = G$, $[X, Y; G]$ is the usual commutator subgroup of X by Y, which we shall denote simply by $[X, Y]$. We recall that the *lower central series* of G is defined by the recursive formulae: ${}^0G = G$, ${}^{i+1}G = [{}^iG, G]$. In the next paragraph we shall introduce a generalization of the notion of lower central series.

Let S be a subgroup of the group G, and consider the subgroups ${}_iS = {}_i(S, G)$ defined inductively by the formulae: ${}_0S = S$, ${}_{i+1}S = [{}_iS, G; {}_iS]$. It is clear from this definition that

2.1 ${}_{i+1}S \subseteq {}_iS$.

Furthermore, obvious induction arguments give

2.2 ${}^iS \subseteq {}_iS \subseteq {}^iG \cap S$;

2.3 *if α is any endomorphism of G such that $S^\alpha \subseteq S$, then $({}_iS)^\alpha \subseteq {}_iS$.*

It follows in particular that the subgroups ${}_iS$ constitute a descending chain of subgroups of S, each normal in $N(S)$, in which all possible factors are nilpotent, and adjacent factors are abelian. We call this chain the *focal series of S in G*. In case $S = G$, the focal series coincides with the lower central series of G. In case $S = N$ is a normal subgroup of G, the focal series of N in G is the N-central series of G in the sense of Baer [**1**]. Our main concern is with the case in which S is a proper, non-normal subgroup of G.

In this section we establish a number of lemmas concerning focal series, which we shall use in the sequel.

If M and N are normal subgroups of G, then ${}_i(MN) = {}_iM\,{}_iN$; indeed, this is a standard formula of the commutator calculus. We prove

Lemma 2.4. *If the subgroup S of the group G is the direct product $S = A \otimes B$ of two of its subgroups A and B which have relatively prime orders, then ${}_iS = {}_iA \otimes {}_iB$.*

Proof. If the transform s^g of an element s of S by an element g of G is in S, i.e. if the commutator $[s, g]$ is in S, we have $s^g = ab$, with a in A and b in B; and $s = xy$ with x in A and y in B. If we let n denote the order of B, we can find an integer k such that $x = x^{kn} = (xy)^{kn} = s^{kn}$. Hence

$$x^g = g^{-1}(s)^{kn}g = (g^{-1}sg)^{kn} = (s^g)^{kn} = (ab)^{kn} = a^{kn},$$

so that x^g is in A, i.e. $[g, x]$ is an element of A. Similarly $[g, y]$ is in B, hence $[g, s] = [g, xy] = [g, y][g, x]^y$ is an element of ${}_1A \otimes {}_1B$. But ${}_1S$ is generated by such commutators $[g, s]$ so that ${}_1S \subseteq {}_1A \otimes {}_1B$. The reverse inequality is obvious, hence ${}_1S = {}_1A \otimes {}_1B$. Now the lemma follows by an obvious induction.

Many examples show that the formula ${}_i(ST) = {}_iS\,{}_iT$ is not valid for arbitrary pairs of subgroups. For instance

Example 1. Suppose that N is the direct product of two cyclic subgroups $A = \{a\}$ and $B = \{b\}$, each of order p, p a prime. Consider the group G formed by adjoining to N the automorphism c of N defined by $a^c = a$, and $b^c = ab^{-1}$. One verifies easily that (i) c has order 2, (ii) $A = Z(G)$, so that ${}_1A = [A, A] = 1$, and (iii) ${}_1B = 1$. The commutator $[ab, c] = ab^{-2}$ is an element $\neq 1$ of ${}_1N$, hence $1 = {}_1A\,{}_1B \subset {}_1N$. This example shows that the hypothesis that the orders of A and B are relatively prime is needed for Lemma 2.4.

Let us consider furthermore the commutators c_i defined recursively by: $c_1 = [ab, c]$ $c_{i+1} = [c_i, c]$. Set $n_i = (-1)^{i+1}\,2^i$, then by a straightforward induction argument one verifies that $c_i = (ab^{-2})^{n_i}$. Hence c_i is an element of ${}_iN$, and if p is an odd prime, $c_i \neq 1$. Hence in this case, N is not part of the hypercentre of G, for otherwise, since N is normal in G, we would have ${}_iN = 1$ for some i.

LEMMA 2.5. *If N is a normal subgroup of G such that $G = NS$, then ${}_iS \subseteq {}^iS[N \cap S]$.*

Proof. We prove by induction on i that ${}_iS \subseteq {}^iSN$. Since ${}_0S = {}^0S = S$, this statement is true for $i = 0$. Let us assume that it is true for some i, $0 < i$. ${}_{i+1}S$ is generated by commutators $c = [s, g]$, with s in ${}_iS$, g in G. Since $G = NS$, $g = nt$, with n in N, t in S. Hence, since N is a normal subgroup of G, $c = [s, g] = [s, nt] = [s, t][s, n]^t \equiv [s, t]$ modulo N. By the induction hypothesis, $s = mr$, with m in N and r in ${}^iS$. Hence $[s, t] = [mr, t] = [m, t]^r\,[r, t] \equiv [r, t]$ modulo N, again using the fact that N is a normal subgroup of G. But r is in ${}^iS$ and t is in S, hence $[r, t]$ is in ${}^{i+1}S$. Now we have proved that c is in ${}^{i+1}SN$, from which it follows that ${}_{i+1}S \subseteq {}^{i+1}SN$. This completes the induction. Thus ${}_iS \subseteq {}^iSN \cap S = {}^iS[N \cap S]$, by Dedekind's law.

PROPOSITION 2.6. *If N is a normal subgroup and S a subgroup of G such that $G = NS$ and ${}_iS = N \cap S$, then ${}_iS = {}^iG \cap S$, and more generally, for $j < i$, ${}_jS = {}^jS[{}^iG \cap S]$.*

Proof. Since ${}_iS$ is a normal subgroup of G and ${}^iS \subseteq {}_iS \subseteq {}^iG$, we have by Lemma 1.1 (b) that ${}^iG \cap S = N \cap S$, and hence that ${}_iS = {}^iG \cap S$. Using this fact together with Lemma 2.5, we have, for $j < i$,

$${}_jS \subseteq {}^jS[N \cap S] = {}^jS[{}^iG \cap S] = {}^jS\,{}_iS \subseteq {}_jS.$$

Hence ${}_jS = {}^jS[{}^iG \cap S]$, which proves the proposition.

That the formula ${}_iS = {}^iG \cap S$ is not universally true can be easily seen. For instance

Example 2. Let S be a subgroup of order p in a p-group G; $S = \{s\}$. If for an element g in G the commutator $[g, s]$ is in S, g induces an automorphism of S of order a divisor of $p - 1$, since S has order p. But, since g has order a power of

p, the order of this automorphism must divide p. Thus g must induce the identity automorphism in S, hence $[g, s] = 1$, so that ${}_1S = 1$. If now S is part of ${}^1G$, $1 = {}_1S \subset {}^1G \cap S$.

We wish to apply certain theorems of Burnside and Grün concerning conjugacy of elements in subgroups. First:

(Burnside.) *Two subsets of a group G which are normalized by a Sylow subgroup S of G (in particular, two elements of the centralizer of S in G) are conjugate in G only if they are conjugate in the normalizer of S in G.*

When we recall the generalized Sylow theorems of P. Hall for solvable groups, we have by an identical argument to that used in proving this result of Burnside (see for example [**9**, p. 139]) that:

If G is a solvable group, and if S is a subgroup of G with order prime to its index in G, then two subsets of G which are normalized by S (in particular, two elements of the centralizer of S in G) are conjugate in G only if they are conjugate in the normalizer of S in G.

Now let S be a p-Sylow subgroup of a group G, and denote by $B(S)$ the weak closure of $Z(S)$ in S, i.e., the subgroup generated by all the conjugates to $Z(S)$ which are contained in S. Following Grün, we call G *p-regular* if each p-Sylow subgroup of G which contains $Z(S)$ also contains $B(S)$. Notice that this definition is independent of the particular p-Sylow subgroup S. For p-regular groups, a result of the type we are interested in is the following theorem of Grün [**6**, §4, Satz 2]:

If S is a p-Sylow subgroup of a p-regular group G, then two subsets of G normalized by $Z(S)$ (in particular, two elements of S) which are conjugate in G are conjugate in the normalizer of $B(S)$ in G.

p-regularity is a generalization of the concept, also due to Grün, of p-normality. A group is called *p-normal* if the centre of a p-Sylow subgroup is the centre of each p-Sylow subgroup in which it is contained. Thus G is p-normal if and only if $Z(S) = B(S)$ for S a p-Sylow subgroup of G.

The applications of the theorems of Burnside and Grün which we have in mind are the following:

LEMMA 2.7. *If S is a Sylow subgroup of the group G, or if S is a subgroup of a solvable group G, with order prime to its index in G, and if H is any subgroup of the centralizer of S in G, then the focal series of H in G coincides with the focal series of H in the normalizer of S in G.*

LEMMA 2.8. *If S is a p-Sylow subgroup of a p-regular group G, then the focal series of S in G coincides with the focal series of S in the normalizer of $B(S)$ in G.*

The proofs of these lemmas are fairly obvious. Suppose for example that S is a Sylow subgroup of G or that the order and index of S in the solvable group G

are relatively prime, and suppose that H is a subgroup of the centralizer $C(S)$ of S in G. We shall prove by induction that ${}_i(H, G) = {}_i(H, N(S))$, where $N(S)$ is the normalizer of S in G. Since ${}_0(H, G) = {}_0(H, N(S)) = H$, the proposition is true for $i = 0$. Assume its validity for some $i > 0$. ${}_{i+1}(H, G)$ is generated by commutators $c = [h, g]$ with c and h in ${}_i(H, G)$, g in G. But $c = h^{-1} h^g$, so that h^g is also in ${}_i(H, G)$. Hence it follows from the above mentioned theorem of Burnside that there exists an element k in $N(S)$ such that $h^g = h^k$, so that $c = [h, g] = [h, k]$. By the induction assumption c and h are in ${}_i(H, N(S))$, hence c is in ${}_{i+1}(H, N(S))$. Thus ${}_{i+1}(H, G) \subseteq {}_{i+1}(H, N(S))$, which completes the induction, since the opposite inequality is clear. This proves Lemma 2.7. The proof of Lemma 2.8 follows exactly similar lines, so we omit it.

3. Focal chains: a divisibility theorem. We define a *focal chain* in a group G to be a set of subgroups S_i of G subject to the conditions

(1) $S_{i+1} \subseteq S_i$,

(2) ${}_1(S_i, G) \subseteq S_{i+1}$.

Notice that since $[S_i, S_i] \subseteq {}_1(S_i, G)$, S_{i+1} is a normal subgroup of S_i, and S_i/S_{i+1} is abelian.

If S is a subgroup of G, then the focal series of S is the minimal focal chain in G beginning with S, that is, the focal series of S is a focal chain beginning with S, and if the subgroups S_i form another such focal chain, then ${}_iS \subseteq S_i$. The intersection of S with the lower central series of G, i.e. the set of subgroups $S \cap {}^iG$, is a second example of a focal chain beginning with S.

If the subgroups S_i form a focal chain in G, and if H is a subgroup of G, then the subgroups $S_i \cap H$ also form a focal chain in G.

In case there exists a focal chain in G which begins with the subgroup S of G and terminates with the subgroup H of S, we shall say that H is *chained* to S in G. If H is chained to S, then ${}_iS \subseteq H$ for some i. That the converse is not true is easily seen; for instance:

Example 3. Let the group N be the direct product of two cyclic groups $\{a\}$ and $\{b\}$ each of order p, p a prime, and form the group G by adjoining to N the automorphism c of N defined by: $a^c = a$, $b^c = ab$. Then ${}^1G = {}_1N = \{a\}$, but $\{a\} = Z(G)$, hence ${}^2G = {}_2N = 1$. Thus ${}_2N \subseteq \{b\}$, but there exists no focal chain from N to $\{b\}$ since ${}_1N$ not $\subseteq \{b\}$.

THEOREM 3.1. *Let S be a subgroup of the group G, and assume that the subgroup H of S is chained to S in G. Assume moreover that the set π of primes contains all the prime divisors of the index $[S: H]$. Then every prime divisor of $[P(\pi) \cap S: P(\pi) \cap H]$ divides $[P(\pi): P(\pi) \cap S]$.*

To prove the theorem we need two lemmas.

LEMMA 3.2. *If S is a subgroup of the group G, then there exists a homomorphism σ of G into $S/{}_1S$ such that $x^\sigma = {}_1S\, x^{[G:S]}$ for x in S.*

Proof. The transfer of Schur [**8**; **9**, chap. V] is a homomorphism τ of G into $S/[S, S]$ which may be computed according to the formula

$$x^\tau = [S, S]\prod_{i=1}^{r} t_i^{-1}x^{f_i}\, t_i, \qquad x \in G,$$

where $t_i^{-1}x^{f_i}\, t_i$ is an element of S, and $f_1 + \ldots + f_r = [G\colon S]$. Since $[S, S] \subseteq {}_1S \subseteq S$, there exists a natural homomorphism ν of $S/[S, S]$ onto $S/{}_1S$. Since the factors $t_i^{-1}x^{f_i}\, t_i$ are in S, it follows that for x in S,

$$t_i^{-1}x^{f_i}\, t_i = x^{f_i}[x^{f_i}, t_i] \equiv x^{f_i} \qquad \text{mod } {}_1S.$$

Hence if we set $\sigma = \tau\nu$ we have for each element x in S

$$x^\sigma = {}_1S\prod_{i=1}^{r} x^{f_i} = {}_1S\, x^{f_1+f_2+\ldots+f_r} = {}_1S\, x^{[G:S]}.$$

Thus σ has the required property, which proves the lemma.

Lemma 3.3. *Let H be a normal subgroup of the subgroup S of G, and let π be a set of primes which contains all the prime divisors of* $[S\colon H]$. *Then if we set $A = P(\pi) \cap S$ and $B = P(\pi) \cap H$ we have*

$$A^{[P(\pi):A]} \subseteq {}_1(A, P(\pi))B.$$

Proof. By Lemma 3.2 there exists a homomorphism λ of $P(\pi)$ into $A/{}_1(A, P(\pi))B$ such that $x^\lambda = {}_1(A, P(\pi))B\, x^{[P(\pi):A]}$ for x in A. Since π contains all the prime divisors of $[S\colon H]$, π contains all the prime divisors of the divisor

$$[A\colon B] = [P(\pi) \cap S\colon P(\pi) \cap H]$$

of $[S\colon H]$. Hence, since $P(\pi)$ is generated by elements whose orders have no prime divisors in π, it follows that $\lambda = 0$. But this implies that $A^{[P(\pi):A]} \subseteq {}_1(A, P(\pi))B$, which is the desired result.

On the basis of Lemma 3.3 it is easy to give a

Proof of Theorem 3.1. Assume that there exists a focal chain $S_i (i = 0, 1, \ldots, k)$ in G, such that $S_0 = S$ and $S_k = H$. Assume also that the set π of primes contains all the prime divisors of $[S\colon H]$. Then the subgroups $T_i = P(\pi) \cap S_i$ form a focal chain with $T_0 = P(\pi) \cap S$, $T_k = P(\pi) \cap H$. Thus T_{i+1} is a normal subgroup of T_i, and π contains all the prime divisors of the divisor

$$[T_i\colon T_{i+1}] = [P(\pi) \cap S_i\colon P(\pi) \cap S_{i+1}]$$

of $[S\colon H]$. Hence by Lemma 3.3, we have

$$T_i^{[P(\pi):T_i]} \subseteq {}_1(T_i, P(\pi))\, T_{i+1} = T_{i+1}. \tag{3.4}$$

Now we shall prove by induction on i that every prime divisor of $[P(\pi)\colon T_i]$ divides $[P(\pi)\colon T_0]$. This statement is clearly true for $i = 0$; assume then that it is true for some i, $0 \leqslant i < k$. Since $T_{i+1} \subseteq T_i \subseteq P(\pi)$, we have

$$[P(\pi)\colon T_{i+1}] = [P(\pi)\colon T_i][T_i\colon T_{i+1}],$$

and by 3.4, every prime divisor of $[T_i: T_{i+1}]$ divides $[P(\pi): T_i]$. Now it follows from the induction hypothesis that every prime divisor of $[P(\pi): T_{i+1}]$ divides $[P(\pi): T_0]$, completing the induction. The theorem follows.

Remark 1. If H is a subgroup of a subgroup S of G, then for π a set of primes, the statements
(i) π contains all the prime divisors of $[S: H]$,
(ii) π contains all the prime divisors of $[P(\pi) \cap S: P(\pi) \cap H]$,
are equivalent.

For
$$\begin{aligned}[S: H] &= [S: (P(\pi) \cap S)\, H][(P(\pi) \cap S)\, H: H] \\ &= [P(\pi)\, S: P(\pi)\, H][P(\pi) \cap S: P(\pi) \cap H],\end{aligned}$$

and $[P(\pi)S: P(\pi)\, H]$ divides the order of the π-group $G/P(\pi)$.

Remark 2. In case ${}_iS = 1$ for some i, Theorem 3.1 implies that for $\pi =$ the totality of prime divisors of the order of S, every prime divisor of $P(\pi) \cap S$ divides $[P(\pi): P(\pi) \cap S]$. That this does not imply $P(\pi) \cap S = 1$ is shown by the following example of a group which is generated by elements of order prime to the prime number p, but which contains a centre element of order p.

Example 4. Let A be the direct product of two cyclic groups $\{a\}$ and $\{b\}$ each of order p, p an odd prime. Let i be an integer $1 < i < p$, then the equations: $a^c = a$, $b^c = b^i$, and $a^d = a$, $(ab)^d = (ab)^i$ define automorphisms c and d of A, which have the same order n, equal to the multiplicative order of i modulo p. In particular, n is prime to p. In the finite group G, obtained by adjoining to A the group of automorphisms U of A generated by c and d, a is a centre element of order p, and c and d have order n prime to p. Furthermore the element bc has order prime to p, for $(bc)^n = b^m c^n = b^m$, where

$$m = 1 + i + i^2 + \ldots + i^{n-1} = (i^n - 1)(i - 1)^{-1},$$

since $i - 1$ is prime to p. Hence p divides m, so that $(bc)^n = b^m = 1$. Similarly the order of $(ab)d$ is prime to p. But the elements c, d, bc, abd generate G. Thus G is generated by elements of order prime to p, and contains a centre element of order p.

COROLLARY 3.5. *If the subgroup H of S is chained to S, and if π is a set of primes which contains all the prime divisors of $[S: H]$, but none of $[G: S]$, then $G = P(\pi)S$ and $P(\pi) \cap S \subseteq H$.*

Proof. Since π contains no prime divisors of $[G: S]$, $G = P(\pi)S$ by 1.3. Since furthermore π contains all the prime divisors of $[S: H]$, this index is relatively prime to

$$[G: S] = [P(\pi)S: S] = [P(\pi): P(\pi) \cap S].$$

Hence $P(\pi) \cap S = P(\pi) \cap H$ by Theorem 3.1. This proves the Corollary.

Now we obtain, as an application of Corollary 3.5, Proposition 1.8, Corollary 1.9, and Proposition 2.6.

THEOREM 3.6. *Assume that the normal subgroup H of the subgroup S of G satisfies the condition*

3.61 *H is chained to S in G, ${}^iS \subseteq H \subseteq {}^iG$, and $[G: S]$ and $[S: H]$ are relatively prime.*

Then if π is any set of primes which contains all the prime divisors of $[S: H]$, but none of $[G: S]$ (for example, if π is the totality of prime divisors of $[S: H]$) we have

(i) *$N = P(\pi)H$ is a normal subgroup of G such that $G = NS$ and $H = N \cap S$.*

(ii) *For N a normal subgroup of G, the following are equivalent statements:*

(a) *$G = NS$ and $H = N \cap S$;*

(b) *$N = P(\pi)H = G^i(\pi)$;*

(c) *$N = G^i(\sigma)$, σ a set of primes which contains all the prime divisors of $[S: H]$, but none of $[G: {}^iGS]$.*

(iii) *$H = {}^iG \cap S$, and in case $H = {}_iS$, we have ${}_jS = {}^jS[{}^iG \cap S]$ for $j < i$.*

Remark 3. If the indices $[G: S]$ and $[S: {}_kS]$ are relatively prime for some k, and if $i < k$, then the hypotheses of Theorem 3.6 are satisfied for $H = {}_iS$ and $\pi =$ the totality of prime divisors of $[S: {}_kS]$. Thus, for example, if S is a p-Sylow subgroup of G, then for each i, all the conclusions of Theorem 3.6 are valid with $H = {}_iS$ and $\pi = p$.

Remark 4. Simple examples show that the condition that the indices $[G: S]$ and $[S: H]$ be relatively prime is not necessary for the conclusion (i) of Theorem 3.6. Indeed, suppose that A is a p-group, and that B is a group which is generated by elements of order prime to the prime number p, but that the order of B is divisible by p. Set $G = A \otimes B$, then $B = P(p)$, and it is clear that for $S = A$, $H = {}_iA$, and $\pi = p$, (i) is satisfied for each i.

For a second example, suppose that U is a group which is generated by elements whose orders are not divisible by p, but such that p divides the order of U, and U admits an automorphism a of order a power of p—for instance, if p divides the order of $U/Z(U)$, U has an inner automorphism of order p. Form the group G by adjoining a to U. Then $U = P(p)$, and for $S =$ the cyclic subgroup of G generated by a, $H = {}_iS$, and $\pi = p$; (i) is fulfilled for each i.

COROLLARY 3.7. *Let H and K be respectively normal subgroups of the subgroups S and T of G, subject to the condition* 3.61 *of Theorem* 3.6. *Assume furthermore that the totalities of prime divisors of* $[S: H]$ *and* $[T: K]$ *coincide. Then if H is conjugate to K, S is conjugate to T.*

Proof. If there exists x such that $H = K^x$, set $R = T^x$. Then H is a normal subgroup of R, and satisfies the condition 3.61 with respect to R. Moreover, since $[R: H] = [T: K]$, the totalities of prime divisors of $[S: H]$ and $[R: H]$ coincide. If π denotes this totality, we have by Theorem 3.6 that $N = P(\pi)H$ is a normal subgroup of G such that $G = NS = NR$, and $H = N \cap S = N \cap R$.

Both S and R are part of the normalizer $N(H)$. Set $M = N \cap N(H)$, then M is a normal subgroup of $N(H)$, and $H \subseteq M$. For X any subgroup of G which contains H and is part of $N(H)$, set $X^* = X/H$. Then S^* and R^* are nilpotent representative subgroups of the normal subgroup M^* of $N(H)^*$. Since $[N(H){:}\,M] = [G{:}\,N] = [S{:}\,H]$, and $[M{:}\,H]$ divides $[N{:}\,H] = [G{:}\,S]$, M^* has order prime to its index in $N(H)^*$. Hence it follows [**9**, p. 132] that S^* is conjugate to R^* in $N(H)^*$, and hence that S is conjugate to R in $N(H)$. Now the corollary is proved, since the relation of conjugacy is transitive.

As a further application of Theorem 3.6 we shall prove the following generalization of a theorem of Grün [**4**; **6**].

Corollary 3.8. *If S is a p-Sylow subgroup of a p-regular group G, and if N_B is the normalizer of the weak closure $B = B(S)$ of $Z(S)$ in S* (see §2 for the necessary definitions) *then $S \cap {}^iG = S \cap {}^iN_B$, and if we denote this subgroup by $S(i)$ we have the isomorphisms*

$$S/S(i) \simeq N_B/N_B{}^i(p) \simeq G/G^i(p).$$

Proof. By Theorem 3.6 we have the equations

$$G = G^i(p)S,\ {}_i(S, G) = G^i(p) \cap S = {}^iG \cap S,$$

and

$$N_B = N_B{}^i(p)S,\ {}_i(S, N_B) = N_B{}^i(p) \cap S = {}^iN_B \cap S.$$

By Lemma 2.8, ${}_i(S, G) = {}_i(S, N_B)$. Now the Corollary follows.

4. Hyperfocality. In analogy with the terminology of centre, hypercentre, etc., we shall refer to a subgroup S of the group G as a *focal* subgroup of G if ${}_1S = 1$, and more generally, as a *hyperfocal* subgroup of G if ${}_kS = 1$ for some k. Thus a focal subgroup is abelian, and a hyperfocal subgroup is nilpotent.

The hypercentre $H(G)$ of G may be defined in several equivalent ways. We recall only, that using our terminology,

$H(G)$ = *the product of all normal, hyperfocal subgroups of G*,

from which it is clear that the hypercentre, and hence any hypercentral subgroup, is hyperfocal. We have already discussed other examples of hyperfocal subgroups. Example 1 of §2 shows that a (hyper) focal subgroup need not be part of the hypercentre. It furthermore shows that the product of two (hyper) focal subgroups need not be hyperfocal, and indeed that a (hyper) focal subgroup need not be hyperfocal modulo the centre. In Example 2 of §2 we proved that every subgroup of order p in a p-group is hyperfocal, which shows in particular that a focal subgroup which is part of the hypercentre need not be part of the centre.

A further indication of the relation between the concepts of hypercentrality and hyperfocality is given by the following strengthening of a theorem of Baer [**2**].

PROPOSITION 4.1. *If p is a prime, and if the subgroup S of the group G is a p-group, then the following conditions are equivalent:*
(i) $S \subseteq H(G)$.
(ii) $P(p) \subseteq C(S)$.
(iii) $P(p) \subseteq N(S)$, *and* $S \cap P(p)$ *hyperfocal in* $P(p)$.

Proof. The equivalence of the conditions (i) and (ii) is the theorem of Baer. If $P(p) \subseteq C(S)$, then $S \cap P(p) \subseteq Z(P(p))$, hence, since $C(S) \subseteq N(S)$, (ii) implies (iii).

To complete the proof, we shall show that (iii) implies (ii). Assume (iii). Then $S \cap P(p)$ is a normal hyperfocal subgroup of $P(p)$, and hence belongs to the hypercentre of $P(p)$. It follows that since (i) implies (ii), $S \cap P(p)$ is part of the centre of $P(p)$. Now let x be an element of G whose order is prime to p (by definition $P(p)$ is generated by such elements) and let s be an element of S. Since x is in $P(p)$, the commutator $c = [x, s]$ is in $P(p)$, and since $P(p) \subseteq N(S)$, c is in S. Thus x is in $S \cap P(p)$, and hence is an element of the centre of $P(p)$. Hence $cx = xc$, from which we prove by a standard method that the order of c divides the order of x. But c is in S, and hence has order a power of p. It follows therefore that $c = 1$. We have proved that (ii) is a consequence of (iii), which completes the proof of the proposition.

Now we come to the problem of characterizing in a group G those subgroups S which are hyperfocal. In order to be able to apply results of §3, we proceed under the assumption that the order of S is prime to its index in G. For the sake of brevity, we shall refer to a subgroup which satisfies this condition as a *P-subgroup*. We begin by proving

PROPOSITION 4.2 (a) *If S is a P-subgroup of a group G and if $Z(S)$ is part of the hypercentre of $N(S)$, then $Z(S)$ is part of the centre of $N(S)$.*

(b) *If S is a Sylow subgroup of G, or if S is a P-subgroup of a solvable group G, then the following conditions are equivalent:*
(i) $Z(S)$ *is focal in* G.
(ii) $Z(S)$ *is hyperfocal in* G.
(iii) $Z(S)$ *is part of the hypercentre of* $N(S)$.
(iv) $Z(S)$ *is part of the centre of* $N(S)$.

Proof. If S is a P-subgroup of G, S is a P-subgroup of $N(S)$. Hence if U is the subgroup which is generated by all the elements of $N(S)$ which have orders prime to the order of S, and hence to the order of $Z(S)$, $N(S) = SU$. If $Z(S)$ is part of the hypercentre of $N(S)$, $Z(S)$ commutes elementwise with U, by the theorem of Baer (Proposition 4.1), and hence in this case $Z(S)$ is part of the centre of $N(S)$. This proves (a).

Assume now that S is a Sylow subgroup of G, or that S is a P-subgroup of the solvable group G. That (i) implies (ii) is trivial. Since $Z(S)$ is normal in $N(S)$, $Z(S)$ is part of the hypercentre of $N(S)$ whenever it is hyperfocal in $N(S)$. But (ii) implies that $Z(S)$ is hyperfocal in $N(S)$, hence (ii) implies (iii). That

(iii) implies (iv) is a consequence of part (a) of the present proposition. Finally we conclude the implication of (i) by (iv) from Lemma 2.7.

THEOREM 4.3. *If S is a P-subgroup of the group G, and if π denotes the totality of prime divisors of the order of S, then the following conditions are equivalent:*

(i) *S is hyperfocal in G.*

(ii) *S is nilpotent, and $1 = P(\pi) \cap S$.*

(iii) *S is nilpotent, and $P(\pi)$ = the totality of elements in G of order prime to the order of S.*

(iv) *S is nilpotent, and there exists a normal subgroup N of G such that $G = NS$ and $1 = N \cap S$.*

(v) *S is nilpotent, and ${}_iS = {}^iS$ for $0 < i$.*

(vi) *S has only the identity element in common with the hypercommutator subgroup of G* (the hypercommutator subgroup of G is defined to be the smallest term of the lower central series of G).

(vii) *S is nilpotent, there exists a subgroup T of G such that ${}_iN(S) = {}_iS \otimes {}_iT$, and ${}_i(S, G) = {}_i(S, N(S))$ for $0 < i$.*

(viii) *S is nilpotent, $N(S) = SC(S)$, and ${}_i(S, G) = {}_i(S, N(S))$ for $0 < i$.*

THEOREM 4.3′. *If S is a Sylow subgroup of G, or if S is a P-subgroup of the solvable group G, then the following are equivalent conditions:*

(i) *S is hyperfocal in G.*

(ix) *S is nilpotent, and ${}^iN(S) = {}_iS_i(C(S), N(S))$ for $0 < i$.*

(x) *S is nilpotent, and ${}^iN(S) = {}_iS_iC(S)$ for $0 < i$.*

(xi) *$Z(S) \subseteq Z(N(S))$, and ${}_kS \subseteq Z(S)$ for some k.*

(xii) *S is part of the hypercentre of $N(S)$, and ${}^iN(S) = {}_iN(S)$ for $0 < i$.*

Remark 5. In case S is a Sylow subgroup of G, we strike out the condition "S is nilpotent" whenever it occurs, and the resulting twelve conditions remain equivalent.

Proof of Theorems 4.3 *and* 4.3′. For convenience, let us first record the following fact:

4.31 *If S is a subgroup, N a normal subgroup of the group G such that $G = NS$ and $1 = N \cap S$, then ${}_iS = {}^iS$.*

For then, using Lemma 2.5, ${}_iS \subseteq {}^iS[N \cap S] = {}^iS$. But ${}^iS \subseteq {}_iS$, hence ${}_iS = {}^iS$.

Let us now collect a number of conclusions resulting from the hypothesis that S is a P-subgroup of G. By Corollary 3.5,

(a) *$G = P(\pi)S$, and $P(\pi) \cap S < {}_iS$ for $0 < i$.*

Furthermore, by Theorem 3.6,

(b) *${}_iS = {}^iG \cap S$, for $0 < i$.*

By a theorem of Schur [9, p. 132] there exists a subgroup U of G such that

(c) $C(S) = [S \cap C(S)] \otimes U = Z(S) \otimes U$

and also a subgroup W of G such that

(d) $N(S) = SW$, $1 = S \cap W$.

Now assume (i), that is, assume that there is a k such that ${}_kS = 1$. Then by (a), $P(\pi) \cap S = 1$, so that (i) implies (ii), since hyperfocal subgroups are nilpotent.

If we assume next that $P(\pi) \cap S = 1$, then the order of $P(\pi)$ divides $[G: S]$, which is by hypothesis prime of the order of S. By definition, $P(\pi)$ contains the totality of elements of G of order prime to the order of S, hence $P(\pi)$is equal to this totality. Conversely, if $P(\pi)$ contains only elements of order prime to the order of S, $1 = P(\pi) \cap S$. Thus (ii) and (iii) are equivalent.

Since S is a P-group, $G = P(\pi)S$, hence if $1 = P(\pi) \cap S$, $N = P(\pi)$ satisfies the condition (iv), that is (ii) implies (iv). Furthermore, if there exists a normal subgroup N of G satisfying (iv), ${}_iS = {}^iS$ by 4.31. Hence (iv) implies (v). That (v) implies (i) is an immediate consequence of the definitions of hyperfocality and nilpotency. We have now proved the equivalence of the conditions (i) through (v).

The equivalence of (i) and (vi) is an immediate consequence of the formula (b), when we recall that the hypercommutator subgroup of G is the smallest term of the lower central series of G.

Next assume the existence of a normal subgroup N of G such that $G = NS$, and $1 = N \cap S$. Then by 4.31, ${}_iS = {}^iS$, and by Dedekind's law, $N(S) = [N(S) \cap N] \otimes S$. Hence by 4.31, ${}_i(S, N(S)) = {}^iS$, so that ${}_iS = {}_i(S, N(S))$. Moreover, $T = N(S) \cap N$ has order prime to the order of S, and hence by Lemma 2.4, ${}_iN(S) = {}_iS \otimes {}_iT$. This proves that (vii) is a consequence of (iv).

In case $i = 0$, condition (vii) implies that $N(S) = S \otimes T$, but this clearly implies that $N(S) = SC(S)$. Hence (vii) implies (viii). Conversely, if we assume that $N(S) = SC(S)$, then by (c),

$$N(S) = S([S \cap C(S)] \otimes U) = S \otimes U.$$

Hence by Lemma 2.4, (viii) implies (vii). Thus (vii) and (viii) are equivalent.

Assuming (vii), we have by 4.31 that ${}_i(S, N(S)) = {}^iS$, and hence that ${}_iS = {}^iS$. This means that (v) is a consequence of (vii). The equivalence of the conditions (i) through (viii) is now established, proving Theorem 4.3.

In proving Theorem 4.3′, we may use the equivalence (i) through (viii). Now assume once more that $N(S) = SC(S)$, then, since S and $C(S)$ are normal subgroups of $N(S)$, ${}^iN(S) = {}_i(S, N(S))\ {}_i(C(S)N(S))$. Hence (viii) implies that ${}^iN(S) = {}_iS\ {}_i(C(S), N(S))$, so that (viii) implies (ix). If we assume that G is solvable, or that S is a Sylow subgroup of the group G, then the equivalence of (ix) and (x) is immediate by Lemma 2.7.

Assume (ix), then in particular $N(S) = SC(S)$, and this condition implies by (c) that S is a direct factor of $N(S)$. Hence by 4.31, ${}_i(S, N(S)) = {}^iS$. Since by (ix), S is nilpotent, there is a k such that $1 = {}^kS = {}_k(S, N(S))$, i.e., S is part of the hypercentre of $N(S)$. Now we conclude by (ix) that ${}_kS \subseteq {}_k(C(S), N(S)) \cap S \subseteq C(S) \cap S = Z(S)$. Furthermore, since S is part of the hypercentre of $N(S)$, so is $Z(S)$. Hence by Proposition 4.2, $Z(S)$ is part of the centre of $N(S)$. Hence (ix) implies (xi).

Now assume that G is solvable, or that S is a Sylow subgroup of the group G, and assume (xi). Then ${}_{k+1}S \subseteq {}_1Z(S)$, and by Lemma 2.7, and condition (xi), ${}_1Z(S) = (Z(S), N(S)) = 1$, hence ${}_{k+1}S = 1$. Thus in case G is solvable, or S is a Sylow subgroup of G, (i) is a consequence of (xi), and then the conditions (ix), (x), and (xi) are equivalent to each other, and to the preceding conditions.

Finally, we must consider the condition (xii). Assume that G is solvable, or that S is a Sylow subgroup. First assume the preceding equivalent conditions. By (i), S is part of the hypercentre of $N(S)$. By (vii), ${}_tN(S) = {}_tS \otimes {}_tT$, and ${}^tN(S) = {}_t(S, N(S)) \otimes {}_t(T, N(S))$, and ${}_t(S, N(S)) = {}_tS$. Since furthermore $T \subseteq C(S)$, we have by Lemma 2.7 that ${}_t(T, N(S)) = {}_tT$. Hence ${}_tN(S) = {}^tN(S)$. Thus in this case (xii) is a consequence of the preceding equivalent conditions.

Now assume (xii). By (d), there exists a subgroup W of G such that $N(S) = SW$ and $1 = S \cap W$. Since S is part of the hypercentre of $N(S)$, it follows from Proposition 4.1 that $W \subseteq C(S)$, and hence $N(S) = S \otimes W$. Hence ${}^tN(S) = {}^tS \otimes {}^tW$, and by Lemma 2.4, ${}_tN(S) = {}_tS \otimes {}_tW$. Since ${}^tN(S) = {}_tN(S)$, and since ${}^tS \subseteq {}_tS$, ${}^tW \subseteq {}_tW$, we conclude that ${}_tS = {}^tS$. Hence, since a subgroup of the hypercentre is nilpotent, (xii) implies (v). This completes the proof of Theorem 4.3′.

It should be remarked that the subgroup N of condition (iv) is uniquely determined,

$N = P(\pi) =$ *the totality of elements of G with orders prime to the order of S.*

COROLLARY 4.4. *For a P-subgroup S of the group G, the following four conditions are equivalent:*

(i) *S is a focal subgroup of G.*
(ii) *S is an abelian hyperfocal subgroup of G.*
(iii) *S is abelian, and there exists a normal subgroup N of G such that $G = NS$ and $1 = N \cap S$.*
(iv) $S \cap [G, G] = 1$.

If S is a Sylow subgroup of G, or if G is solvable, the following condition is equivalent to each of the preceding conditions.

(v) $N(S) = C(S)$.

Proof. The implication of (ii) by (i) is clear, and (iii) follows from (ii) by Theorem 4.3. If (iii) is true, then G/N is isomorphic with the abelian group S, and hence is abelian. Thus $[G, G] \subseteq N$, so that $1 = S \cap [G, G]$. But then $1 = S \cap [G, G] \supseteq {}_1S$, so that ${}_1S = 1$. Thus (iii) implies (iv) and (iv) implies (i), which proves the equivalence of (i) through (iv).

In case S is abelian, (v) is equivalent with condition (xi) of Theorem 4.3′ and (v) implies that S is abelian. Hence, if we assume that S is a Sylow subgroup of G or that G is solvable, we have the equivalence of (v) with the preceding conditions by Theorem 4.3′, which completes the proof.

The implication of (iii) by (v) is the classical theorem of Burnside.

We also mention the following

Corollary 4.5. *If S is a P-subgroup of the group G, the following are equivalent conditions:*

(i) *S is a normal hyperfocal subgroup of G.*
(ii) *S is part of the hypercentre of G.*
(iii) *S is a nilpotent direct factor of G.*
(iv) *S is nilpotent, and $G = SC(S)$.*

This corollary is easily verified by reference to Theorem 4.3 when we remember that a normal hyperfocal subgroup is part of the hypercentre, and notice that each of the conditions (ii) and (iv) imply that S is a normal subgroup of G.

Now it is easy to construct further examples of hyperfocal subgroups which are not part of the hypercentre. For instance, in the group G formed by adjoining to the group A an automorphism a of order prime to the order of A, the cyclic group Z generated by a is a representative subgroup for the normal subgroup A, but is not a direct factor. Hence by Corollary 4.4, Z is focal in G, but by Corollary 4.5, Z is not part of the hypercentre.

Theorem 4.6. *Suppose that the order of the group G is of the form, mn, with m and n relatively prime. Then*

(a) *The following two conditions are equivalent:*
(i) *G contains a hyperfocal subgroup of order m;*
(ii) *a p-Sylow subgroup of G is hyperfocal in G for each prime divisor p of m.*

(b) *If G contains a hyperfocal subgroup S of order m, any subgroup of order m is conjugate to S, and any subgroup of order a divisor of m is contained in a subgroup of order m, and hence is hyperfocal in G.*

Proof. If the prime p divides m, and if S is a subgroup of G of order m, then S contains a p-Sylow subgroup P of G. If S is hyperfocal in G then so is P. Hence (i) implies (ii). Let us assume (ii), then if p^a is the highest power of the prime p which divides m, there exists by Theorem 4.3 a normal subgroup N_p of G such that $[G: N_p] = p^a$. The intersection N for all primes p of the subgroups N_p is a normal subgroup of G, $[G: N] = m$, and G/N is nilpotent. Now it follows by the theorem of Schur that there exists a subgroup S of G of order m, i.e. such that $G = NS$ and $1 = N \cap S$. Since S is isomorphic with G/N, S is nilpotent, and it therefore follows from Theorem 4.3 that S is a hyperfocal subgroup of G. This proves (a).

If now S is any hyperfocal subgroup of G of order m, there exists by Theorem 4.3 a normal subgroup N of G such that $G = NS$ and $1 = N \cap S$. S is nilpotent, and hence G/N is nilpotent. That any subgroup of order m is conjugate to S now follows from a well-known theorem [**9**, p. 132]. Suppose that the order of the subgroup T of G divides m, and consider the subgroup $H = NT$ of G. Since $G = NS$, we have by Dedekind's law that $H = N[H \cap S]$; and H/N is nilpotent, as a subgroup of the nilpotent group G/N. Thus, by the theorem just quoted, there exists an element x in H such that $T = [H \cap S]^x = H \cap S^x$, so that $T \subseteq S^x$. Since S is hyperfocal, so is S^x, and hence, so is T. This completes the proof.

Taking $m =$ the order of G in the preceding proposition, we have

Corollary 4.7. *A group G is nilpotent if and only if for each prime p there exists a p-Sylow subgroup of G which is hyperfocal.*

This corollary also follows from the fact that for S a p-Sylow subgroup of G, ${}_pS = {}^pG \cap S$.

Corollary 4.8. *A subgroup S of a group G is hyperfocal in G if and only if for each prime p there exists a p-Sylow subgroup of S which is hyperfocal in G.*

Proof. That a hyperfocal subgroup of G has the above mentioned property is clear. Assume that S is a subgroup of G which, for each prime, has a Sylow subgroup hyperfocal in G. Then the same is true in S, hence S is nilpotent by Corollary 4.7. Thus S has only one p-Sylow subgroup for each prime p, and is the direct product of these. Now the result follows from Lemma 2.4.

The study of hyperfocal subgroups is now reduced to the study of primary hyperfocal subgroups.

5. A theorem of Grün. The methods of §3 admit of further application, leading in particular to a theorem of Grün. Let S be a subgroup of the group G, and let H be a subgroup of S. Let π denote the totality of prime divisors of $[S:H]$, and set $A = P(\pi) \cap S$, $B = P(\pi) \cap H$. Following Grün, we introduce the subgroup

$S_* =$ the (normal) subgroup of S which is generated by all the intersections $S \cap S^g \neq S$, for g in G.

Theorem 5.1. *If H is a normal subgroup of S, if the indices $[G:S]$ and $[S:H]$ are relatively prime, and if $S_* \subseteq H$, then*

(i) $G = P(\pi)S$.

(ii) $N = P(\pi)H$ *is a normal subgroup of* G.

(iii) $A = [A, N(A \text{ in } P(\pi))]B$.

Proof. Conditions (i) follows from 1.3, and hence (ii) is a consequence of Lemma 1.1. By Lemma 3.3, we have $A^{[P(\pi):A]} \subseteq {}_1(A, P(\pi))B$. But $[P(\pi):A] = [G:S]$ is relatively prime to the divisor $[A:B]$ of $[S:H]$, hence $A = A^{[P(\pi):A]}B$. We conclude that $A = {}_1(A, P(\pi))B$. The subgroup ${}_1(A, P(\pi))B$ is generated by commutators $c = [a, x]$, with c and a in A, x in $P(\pi)$. Since a and c are in A, a^x is in A, hence since $A \subseteq S$, a and a^x are in $S \cap S^x$. If both a and a^x are in S_*, c is in $S_* \cap P(\pi)$. If either a or a^x is not in S_*, it follows that $S = S^x$, that is, x is in

$$N(S \text{ in } G) \cap P(\pi) \subseteq N(A \text{ in } P(\pi)),$$

and hence c is in $[A, N(A \text{ in } P(\pi))]$. This proves that

$${}_1(A, P(\pi)) = [A, N(A \text{ in } P(\pi))][S_1 \cap P(\pi)].$$

Thus

$$A = {}_1(A, P(\pi))B = [A, N(A \text{ in } P(\pi))][S_* \cap P(\pi)]B = [A, N(A \text{ in } P(\pi))]B,$$

since $S_* \cap P(\pi) \subseteq H \cap P(\pi) = B$. Thus (iii) is proved.

COROLLARY 5.2. *If* $S = N(S \text{ in } G)$, *and* $S_* \subseteq H$, *then* (i) *and* (ii) *of Theorem* 5.1 *hold, together with*
(iii′) $A = [A, A]B$, *that is*, A/B *is perfect.*

Proof. First we have

5.21 (Grün) $S = N(S \text{ in } G)$ *implies that* $[G: S]$ *is prime to* $[S: S_*]$.

For suppose that P is a p-Sylow subgroup of S which is not part of S_*. Then there exists a p-Sylow subgroup P^* of G which contains P. If $P \subset P^*$, then $P \subset N(P \text{ in } P^*)$. But $S \cap P^* = P$, hence there is an element x in $N(P \text{ in } P^*)$ which is not in S. Thus $P = P^x$ is part of $S \cap S^x$, which implies that x is in $N(S \text{ in } G) = S$. Since this is impossible, $P = P^*$, which proves 5.21.

Hence, since we have assumed that $S_* \subseteq H$, the indices $[G: S]$ and $[S: H]$ are relatively prime. Thus we conclude (i) and (ii) together with $A = [A, N(A \text{ in } P(\pi))]B$ from Theorem 5.1. Since $S_1 \subseteq H$,

$$P(\pi) \cap S_* \subseteq P(\pi) \cap H = B.$$

In case $P(\pi) \cap S_* = A$, we have $A = B$, so that (iii′) follows in this case. If on the other hand $P(\pi) \cap S_* \subset A$, and if x is an element of $N(A \text{ in } P(\pi))$, $A = A^x \subseteq S \cap S^x$, and hence x is in $N(S \text{ in } G)$. Hence

$$N(A \text{ in } P(\pi)) \subseteq N(S \text{ in } G) \cap P(\pi) = S \cap P(\pi) = A,$$

and (iii′) follows in this case.

Corollary 5.2 has as an immediate consequence the following result of Grün [5]:

COROLLARY 5.3. *If S is a subgroup of the group G such that* $S = N(S \text{ in } G)$, *if the normal subgroup H of S contains every intersection $S \cap S^g$ for g not in S, and if S/H is solvable, then there exists a normal subgroup N of G such that $G = NS$ and $H = N \cap S$.*

Since $S/{}_\dagger S$ is nilpotent, Corollary 5.3 implies that the condition

5.31 $S = N(S \text{ in } G)$, *and* $S_* \subseteq {}_\dagger S$

is sufficient for the existence of a normal subgroup N of G such that $G = NS$ and ${}_\dagger S = N \cap S$. This criterion is subsumed by Theorem 3.6, since 5.31 implies that $[G: S]$ and $[S: {}_\dagger S]$ are relatively prime.

Added in proof. Since this paper was submitted for publication, Professor R. Brauer has communicated to the author theorems concerning the subgroup denoted here by ${}_1S = {}_1(S, G)$ which he obtained as applications of his profound

characterization of group characters. These theorems are contained in our results or are easily obtained by our methods. They appear in Professor Brauer's paper, *A characterization of the characters of groups of finite order*, Annals of Mathematics, *57* (1953), 357-377.

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THE FC-CHAIN OF A GROUP

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1. Introduction. Baer [2] and Neumann [5] have discussed groups in which there is a limitation on the number of conjugates which an element may have. For a given group G, let H_1 be the set of all elements of G which have only a finite number of conjugates in G, let H_2 be the set of those elements of G, the conjugates of each of which lie in only a finite number of cosets of H_1 in G; and in this fashion define $H_3, H_4, \ldots$. We shall show that the H_i are strictly characteristic subgroups of G. The result of Neumann which states that the derivative of G is periodic if $G = H_1$ (that is, if G is a so-called FC-group), and that, in this case, the periodic elements of G form a subgroup reappears in the form that the division hull of H_i in H_{i+1} is a subgroup L_{i+1} such that H_{i+1}/L_{i+1} is abelian. The commutator quotient $H_i \div H_{i+1}$ turns out to be the cross-cut of some collection of subgroups of finite index in G, generalizing a result of Baer [2] on the centralizer of H_1 in G. Hall [6, p. 114] has proved a strict inclusion theorem on the intersections of some subgroups with the ascending central series. A related result is established for the FC-chain $\{H_i\}$. The concept of FC-nilpotency is introduced ($G = H_n$ for some n), and the relation of FC-nilpotency of a factor group of G to the nilpotency of G itself is discussed. We shall prove that the group of automorphisms of a non-trivial, complete centreless group has no non-trivial FC-chain.

2. The FC-chain. Let G be a non-trivial group, and let $H_1 = H_1(G)$ be the set of all $g \in G$, each of which has only a finite number of conjugates in G. By Baer [2], H_1 is a characteristic subgroup of G. Indeed it is more; for, let f be an endomorphism of G where $f(G) = G$, and let $x \in H_1$ have the property that $f(x)$ has more than a finite number of distinct conjugates in G. If $\{r_i^{-1}f(x)r_i\}$ $(i = 1, 2, 3, \ldots)$ is a countable subset of the set of distinct conjugates of $f(x)$, the fact that $f(G) = G$ implies the existence of a set $\{s_i\}$, $s_i \in G$, $f(s_i) = r_i$, so that the $f(s_i^{-1}xs_i)$ are all different, whence the $s_i^{-1}xs_i$ are distinct. But this is a contradiction, so that $f(H_1) \subset H_1$, and H_1 is strictly characteristic.

Let $H_0 = H_0(G)$ be the subgroup of G consisting of e, the identity element of G, alone. Suppose that $H_n = H_n(G)$ has been defined as a suitable normal subgroup of G. Then form $H_1(G/H_n(G))$ and construct its complete inverse image $H_{n+1}(G)$ in G under the natural mapping with kernel $H_n(G)$ which carries G onto $G/H_n(G)$. It is clear that $H_{n+1}(G)$ is a normal subgroup of G and that $H_{n+1}(G)/H_n(G)$ is isomorphic to $H_1(G/H_n(G))$. Thus, inductively, we have fashioned the FC-*chain* of normal subgroups $\{H_j(G)\}$ $(j = 0, 1, 2, \ldots)$ of a group

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G. For all such j, $H_j(G) \subset H_{j+1}(G)$. Moreover, each $H_i = H_i(G)$ is strictly characteristic in G. This statement is true for $i = 0$, and, by the above, it is true for $i = 1$. Now let f be any endomorphism of G for which $f(G) = G$, and suppose that $H_i(G)$ is strictly characteristic in G. If h is an element of $H_{i+1}(G)$, then $hH_i(G)$, as an element of $G/H_i(G)$, has only a finite number of conjugates in the latter group. If, now, $f(h)H_i(G)$ has an infinite number of distinct conjugates, choose a countable set of these, each of the form $z_j^{-1} f(h) z_j H_i$, where each $z_j \in G$ $(j = 1, 2, 3, \ldots)$. Construct elements w_j with $f(w_j) = z_j$. Then the $f(w_j^{-1} hw_j)H_i$ are all distinct. If there exist distinct indices j and k for which

$$(w_j^{-1} hw_j)\, H_i = (w_k^{-1} hw_k)\, H_i,$$

then there exists $h' \in H_i$ with

$$f(w_j^{-1} hw_j) = f(w_k^{-1} hw_k)\, f(h'),$$

where $f(h') \in H_i$ since $f(H_i) \subset H_i$, by the induction assumption. However this implies that

$$f(w_j^{-1} hw_j)\, H_i = f(w_k^{-1} hw_k)\, H_i,$$

a contradiction. Thus, $f(h)H_i$ has only a finite number of conjugates in G/H_i, so that $f(h) \in H_{i+1}$, and the latter subgroup is strictly characteristic.

Let $Z_i = Z_i(G)$ be the ith member of the ascending central series [**6**] of G. Then $Z_i(G) \subset H_i(G)$; for, proceeding inductively, it is clear that $Z_1(G)$, the centre of G, is included in $H_1(G)$. Suppose that $Z_i \subset H_i$ and[1] that $x \in Z_{i+1}$. Then the coset xZ_i is in the centre $Z_1\,(G/Z_i)$ of the group of cosets G/Z_i. Since $Z_i \subset H_i$,

$$xH_i \in Z_1\,(G/H_i) \subset H_1\,(G/H_i) = H_{i+1}/H_i,$$

so that $x \in H_{i+1}$.

Note that $H_i \subset Z_j$ for $i > j$ implies[1] that $Z_i \subset H_i \subset Z_j \subset H_j$, and $Z_i = Z_j = H_i = H_j$, whence both the FC-chain and the ascending central series break off with the same subgroup $Z_j = H_j$. The possibility of course remains that they have so broken off at an index $k < j$. When both the FC-chain and the ascending central series of a group G terminate with the same subgroup $H_j = Z_j$, we say that G has *mutual truncation at index* $\leqslant j$. We replace $\leqslant$ by $=$ if j is the best possible index.

If $x, y \in H_{i+1}$ and if $x \equiv y \bmod Z_{i+1}$, then in the group G/H_i, the elements xH_i and yH_i have the same (finite) number of conjugates.

If $G = H_n(G)$, $G \neq H_{n-1}(G)$, for some positive integer n, we say that G is FC-nilpotent of FC-class n. Hence if G is nilpotent of class m $(G = Z_m(G)$, $G \neq Z_{m-1}(G))$, then G is FC-nilpotent of FC-class $n \leqslant m$.

3. The division hulls. Let K be a subgroup of a group G. By the *division hull* of K in G, $D(K;G)$, we mean the set of all $x \in G$ for which there exist

[1]The author is indebted to the referee for strengthening the argument at this point.
[2]See note 1.

positive integers $n = n(x)$ with $x^n \in K$. If G is abelian or if the set of all $xyx^{-1}y^{-1}$, where $x, y \in D(K; G)$, is included in K, then $D(K; G)$ is a subgroup of G; but, in general, $D(K; G)$ need not be a subgroup of G. If $D(K; G)$ is a subgroup, and if K is admissible under an endomorphism f ($f(K) \subset K$), then $D(K; G)$ is also admissible under f.

The following is easy to prove: If K is a normal subgroup of G and if A^* and B^* are subsets of G/K with respective complete inverse images A and B in G and if $A^* = D(B^*; G/K)$ then $A = D(B; G)$. Likewise in the immediate category is that K normal in G and G/K abelian imply that $D(K; G)$ is a subgroup of G. If every finitely generated subgroup K of G has the property that $D(K; G)$ is a supgroup of G, then $D(L; G)$ is a subgroup of G for every subgroup L of G.

We use the commutator notation of **[6]**. For instance, G' shall mean (G, G), the subgroup of G generated by the commutators of G, the derivative of G; $G'' = (G', G')$. From one of the above results, $D(G'; G)$ is always a subgroup of G. It follows that if K is any normal subgroup of G' for which G'/K is a periodic group, then $D(K; G) = D(G'; G)$. In particular, if G' is a periodic group, $D(G'; G) = P(G)$, the set of periodic elements of G (as we can see by taking K to be the trivial subgroup of G). $P(G)$ is thus **[5]** a subgroup of G whenever $G' \subset P(G)$.

4. Subgroups of H_{i+1}.

LEMMA 1. *$D(H_i; H_{i+1})$ is a normal subgroup L_{i+1} of H_{i+1} ($i = 0, 1, 2, \ldots$).*

Proof. $H_{i+1}/H_i = H_1$ (G/H_i), an FC-group. By a result due to Neumann **[5]**, the set of periodic elements $P = P(H_{i+1}/H_i)$ of H_{i+1}/H_i is a normal subgroup of the latter group. However, $D(H_i; H_{i+1})$ is the complete inverse image in H_{i+1} (under the natural mapping of H_{i+1} onto H_{i+1}/H_i) of P. Consequently, L_{i+1} is a normal subgroup of H_{i+1}.

COROLLARY. *L_{i+1} is a strictly characteristic subgroup of G.*

Proof. If $x \in L_{i+1}$ and if f is an endomorphism of G onto G, then $f(x) \in H_{i+1}$ since H_{i+1} is strictly characteristic. There exists a positive integer n such that $x^n \in H_i$. Hence $f(x^n) = (f(x))^n \in H_i$, since the latter is strictly characteristic. Thus $f(x) \in L_{i+1}$.

Neumann **[5]** has also proved that if G is an FC-group, then $G' \subset P(G)$, the subgroup of periodic elements of G. It follows that, for every FC-group G, $P(G) = D(G'; G)$. Since H'_{i+1} is included in the complete inverse image of $(H_{i+1}/H_i)'$ in H_{i+1}, (under the natural mapping of H_{i+1} onto H_{i+1}/H_i), $H'_{i+1} \subset L_{i+1}$ so that H_{i+1}/L_{i+1} is abelian. If $x \in D(L_{i+1}; H_{i+1})$ then $x^m \in L_{i+1}$ and $x^{mn} \in H_i$ for suitable positive integers m and n. Hence $D(L_{i+1}; H_{i+1}) \subset L_{i+1}$ so that the group H_{i+1}/L_{i+1} is not only abelian but also *torsion-free* in the sense that it has no periodic elements other than its unity.

If G is FC-nilpotent of FC-class n, $n > 1$, then the fact that H_n/L_n is abelian shows that $G' \subset L_n$.

Let ϕ_i be the natural homomorphism of G onto G/H_i. For a subgroup S^* of G/H_i, $\phi_i^{-1}(S^*)$ shall mean the complete inverse image in G of S^* under ϕ_i.

LEMMA 2. *H_{i+1}/L_{i+1} is the trivial group or a direct sum of copies of the group of rationals if and only if to each ordered pair (x, m), where $x \in H_{i+1}$ and m is a positive integer, there corresponds an ordered pair (y, n), where $y \in H_{i+1}$ and n is a positive integer, such that $(xy^m)^n \in H_i$.*

Proof. H_{i+1}/L_{i+1} has the required form if and only if it is *complete* in the (abelian group) sense that

$$xL_{i+1} \in H_{i+1}/L_{i+1} \qquad (x \in H_{i+1})$$

implies, for each positive integer m, the existence of $z = z(m) \in H_{i+1}$ with $(zL_{i+1})^m = xL_{i+1}$. If we let $z^{-1} = y$, the result is immediate.

Let $J_{i+1} = D(H'_{i+1} \cap H_i; H_{i+1})$. $x \in J_{i+1}$ implies $x \in H_{i+1}$ and the existence of a positive integer n for which $x^n \in H'_{i+1} \cap H_i$, so that $x^n \in H_i$ and $x \in L_{i+1}$. But $x^n \in H'_{i+1}$ implies that $x \in D(H'_{i+1}; H_{i+1})$, and $L_{i+1} \subset H_{i+1}$ implies that

$$D(H'_{i+1}; L_{i+1}) \subset D(H'_{i+1}; H_{i+1}),$$

so that both J_{i+1} and $D(H'_{i+1}; L_{i+1})$ are subsets of $D(H'_{i+1}; H_{i+1})$. Conversely, if $x \in D(H'_{i+1}; H_{i+1})$, there exists a positive integer m such that $x^m \in H'_{i+1} \subset L_{i+1}$, and consequently there exists a positive integer n for which $(x^m)^n \in H_i$. This places x in L_{i+1}, hence in $D(H'_{i+1}; L_{i+1})$. Since $x^{mn} \in H'_{i+1}$, $x \in J_{i+1}$, and we have proved that

$$J_{i+1} \equiv D(H'_{i+1} \cap H_i; H_{i+1}) = D(H'_{i+1}; L_{i+1}) = D(H'_{i+1}; H_{i+1}).$$

It is clear that J_{i+1} is a strictly characteristic subgroup of G and that

$$L'_{i+1} \subset H'_{i+1} \subset J_{i+1} \subset L_{i+1} \subset H_{i+1}$$

so that, for instance, $D(J_{i+1}; H_{i+1})$ is a subgroup of G. It is also immediate that the sequences $\{L_{i+1}\}$ and $\{J_{i+1}\}$ are both ascending with i.

5. The commutator quotients. Let S and T be normal subgroups of a group G. Let $S \div T$ be the set of all $x \in G$ which have the property that $(t, x) \in S$ for every $t \in T$. This set is called [**1**] the *commutator quotient* of S by T and is a normal subgroup of G. Let f be an endomorphism of G for which $f(T) = T$ and $f(S) \subset S$. For $x \in S \div T$ and $t \in T$,

$$tf(x)\, t^{-1} f(x^{-1}) = f(uxu^{-1}\, x^{-1}),$$

where $u \in T$; and since $uxu^{-1}\, x^{-1} \in S$, $tf(x)t^{-1} f(x^{-1})$ is likewise in S so that $f(x) \in S \div T$. We have proved that *if S and T are normal subgroups of a group G and if f is any endomorphism of G for which S is admissible and $f(T) = T$, then $S \div T$ is admissible under f.* Moreover it can be shown that $S \div T$ is a characteristic subrgoup of G if both S and T are. Well known is the fact [**1**] that

$S \div T \supset S$ for normal subgroups S and T of G. If S, T and N are normal subgroups of G, it is easy to prove [**1**] that the following are equivalent:

$$\text{(a) } (T, N) \subset S; \qquad \text{(b) } N \subset S \div T; \qquad \text{(c) } T \subset S \div N.$$

The fact that $(N, S \div N) \subset S$ for normal subgroups S and N of G shows that $N \subset S \div (S \div N)$, by the equivalence of (a) and (c). Useful is the result [**1**] that

5.1 $$(N \div G)/N = Z_1\,(G/N)$$

for every normal subgroup N of G.

Since $Z_i \div G = Z_{i+1}$, Z_{i+1} is maximum with respect to being a normal subgroup X for which $(G, X) \subset Z_i$. One would like to have a similar result for the FC-chain, but the facts are otherwise. If we define W_{i+1} by $W_{i+1}\,(G) = H_i(G) \div G$ $(i = 0, 1, 2, \ldots)$ it is easy to see that W_{i+1}, maximum with respect to the property of being a normal subgroup X of G for which $(G, X) \subset H_i$, can be represented by $W_{i+1}\,(G) = \phi_i^{-1}\,(Z_1\,(G/H_i\,(G)))$, upon application of 5.1. W_{i+1}/H_i is abelian, whence $W'_{i+1} \subset H_i \subset W_{i+1}$. Since

$$Z_1\,(G/H_i) \subset H_1\,(G/H_i) \qquad\qquad W_{i+1} \subset H_{i+1}.$$

Since

$$(Z_{i+1}, G) \subset Z_i \subset H_i \qquad\qquad Z_{i+1} \subset W_{i+1}.$$

It is clear that the W_{i+1} form an ascending chain of subgroups of G which "intertwines" with the FC-chain, where $W_1\,(G) = Z_1\,(G)$ and each W_{i+1} is a strictly characteristic subgroup of G. The last remark follows from the fact that $W_{i+1} = H_i \div G$, and that if f is an endomorphism on G onto G, H_i is admissible, so that, by an earlier remark on the admissibility of the commutator quotient, so is W_{i+1} admissible. Note that $H_i \div W_{i+1} = G$, since $W_{i+1} = H_i \div G$ implies $G \subset H_i \div W_{i+1}$.

Let us define $V_{i+1}(G) = H_i\,(G) \div H_{i+1}\,(G)$ $(i = 0, 1, 2, \ldots)$. It follows that $H_{i+1} \subset H_i \div V_{i+1}$. By the symbol $C(H < G)$ for a subgroup H of G we mean [**2**] the centralizer of H in G.

THEOREM 1. *V_{i+1} is maximum with respect to the property of being a normal subgroup X of G for which*

$$(H_{i+1}, X) \subset H_i; \quad W_{i+1} \subset V_{i+1}; \quad V_{i+1} = \phi_i^{-1}(C(H_{i+1}/H_i < G/H_i));$$

and V_{i+1} is the cross-cut of some collection of finite-indexed subgroups of G.

Proof. The first statement derives from the definition of commutator quotient. As a function on the cartesian square of the lattice of normal subgroups of the group G into that same lattice, $X \div Y$ is monotonically increasing in X and monotonically decreasing in Y. Since $V_{i+1} = H_i \div H_{i+1}$ and $W_{i+1} = H_i \div G$, $W_{i+1} \subset V_{i+1}$. As for the third statement, $y \in V_{i+1}$ if and only if $(y, h) \in H_i$ for every $h \in H_{i+1}$. But this is equivalent to the commuting of $\phi_i(y)$ with every $\phi_i(h)$. Since, however, the $\phi_i(h)$ range over all of H_{i+1}/H_i, the third statement

is established. For the last statement, we recall that Baer [2] has showed that, for any group K, $C(H_1(K) < K)$ can be represented as the cross-cut of some collection of finite-indexed subgroups of K. Thus

$$C(H_{i+1}/H_i < G/H_i) = \bigcap_\alpha N_\alpha^*,$$

where each N_α^* is a normal subgroup of finite index in G/H_i. Then

$$\phi_i^{-1}C(H_{i+1}/H_i < G/H_i) = \bigcap_\alpha \phi_i^{-1} N_\alpha^*.$$

Each $\phi_i^{-1} N_\alpha^* = N_\alpha$ is a normal subgroup of G. Since

$$G/N_\alpha \cong (G/H_i)/(N_\alpha/H_i) = (G/H_i)/N_\alpha^*,$$

each N_α has a finite index in G, and the proof is complete.

Since $(H_i. H_i) \subset (V_{i+1}, H_{i+1}) \subset H_i$, $H_i/(V_{i+1}, H_{i+1})$ is abelian, and, by making the normal subgroup X in $(X, H_{i+1}) \subset H_i$ as large as possible, (X, H_{i+1}) itself is moved "above" the derivative H'_i. Likewise $H_i/(G, W_{i+1})$ is abelian. There is, however, a point of dissimilarity between W_{i+1} and V_{i+1}. For normal subgroups X satisfying $W_{i+1} = H_i \div X$, $X = G$ is the obvious maximum which can be obtained. On the other hand, with $V_{i+1} = H_i \div X$, the maximum which X takes on is

$$M_{i+1} = H_i \div V_{i+1} \supset H_i.$$

For $y \in G$, $yH_i = \phi_i(y)$ commutes with every $\phi_i(v) \in V_{i+1}/H_i$ if and only if $y \in M_{i+1}$. Hence

$$M_{i+1} = \phi_i^{-1}\,(C(V_{i+1}/H_i < G/H_i)).$$

Likewise, it is easy to show that

$$V_{i+1} = \phi_i^{-1}\,(C(M_{i+1}H/_i < G/H_i)).$$

Thus, for normal subgroups X of G satisfying $V_{i+1} = H_i \div X$, the maximum is obtained by, essentially, forming centralizers twice from H_{i+1}/H_i.

THEOREM 2. *$W_{i+1} \cap Z_{i+1} = (H_i \cap Z_j) \div G$, so that $W_{i+1} \cap Z_{j+1}$ is maximum with respect to being a normal subgroup X of G for which $(G, X) \subset H_i \cap Z_j$, and*

$$(W_{i+1} \cap Z_{j+1})/(H_i \cap Z_j) = Z_1\,(G/H_i \cap Z_j).$$

Proof. $x \in W_{i+1} \cap Z_{j+1}$ implies that $\phi_i(x)$ and $\phi_i(g)$ commute for every $g \in G$ and that $xgx^{-1}g^{-1} \in Z_j$, since $(G, Z_{j+1}) \subset Z_j$. Thus

$$W_{i+1} \cap Z_{j+1} \subset (H_i \cap Z_j) \div G.$$

Conversely, $(H_i \cap Z_j) \div G \subset H_i \div G$, $Z_j \div G$. But $H_i \div G = W_{i+1}$ and $Z_j \div G = Z_{j+1}$, so that the first statement of the theorem follows. Apply 5.1 as before.

COROLLARY. (a) *If $Z_j \subset H_i$, then $Z_{j+k} \subset W_{i+k}$ $(k = 0, 1, 2, \ldots)$.* (b) *If $H_i \subset Z_j$, then $W_{i+1} \subset Z_{j+1}$.* (c) *If $H_i = Z_j$, then $W_{i+1} = Z_{j+1}$.* (d) *If each*

$W_i = H_i$ $(i = 1, 2, 3, \ldots)$ *then each* $H_i = Z_i$ (*whence each* H_{i+1}/H_i *is abelian, and* G *is* FC-*nilpotent under these conditions if and only if* G *is nilpotent*).

6. A strict inclusion theorem. In the case of the ascending central series, Hall [**6**, p. 114] has proved a strict inclusion theorem. In Theorem 3 below, we shall obtain a similar result for the FC-series.

LEMMA 3. *Let* N *be a normal subgroup of* G *for which* $N \subset W_{i+1}$ *and* $N \not\subset H_i$, *where* $i > 1$. *Then the following inclusions are strict:*

$$N \supset N \cap H_i \supset N \cap H_{i-1}.$$

Proof. $(G, N) \subset N \cap (G, W_{i+1}) \subset N \cap H_i$. If $N \cap H_i \subset N \cap H_{i-1}$, then $(G, N) \subset N \cap H_i$ would imply $(G, N) \subset H_{i-1}$. By the maximum character of W_i, $N \subset W_i \subset H_i$, a contradiction, so that the inclusion $N \cap H_i \supset N \cap H_{i-1}$ is strict. Also, if $N = N \cap H_i$, then $N \subset H_i$, a contradiction, so that the inclusion $N \supset N \cap H_i$ is strict.

THEOREM 3. *If* $Z_{i+1} \not\subset H_i$ *then the following inclusions are strict:*

$$Z_{i+1} \supset Z_{i+1} \cap H_i \supset Z_{i+1} \cap H_{i-1} \supset Z_{i+1} \cap H_{i-2} \supset \ldots \supset Z_{i+1} \cap H_1 \supset (e),$$

where e *is the identity of* G.

Proof. Taking N in Lemma 3 to be Z_{i+1}, we have

$$Z_{i+1} \supset Z_{i+1} \cap H_i \supset Z_{i+1} \cap H_{i-1}$$

with strict inclusions. Since $Z_{i+1} \not\subset H_i$, $Z_{i+1-k} \not\subset H_{i-k}$ (by Corollary (a) of Theorem 2), where $k = 1, 2, 3, \ldots, i$. Suppose that the inclusion

$$Z_{i+1} \cap H_{i+1-k} \supset Z_{i+1} \cap H_{i-k}$$

is strict. Take N in Lemma 3 to be Z_{i+1-k}. Then

$$Z_{i+1-k} \supset Z_{i+1-k} \cap H_{i-k} \supset Z_{i+1-k} \cap H_{i-k-1}$$

with strict inclusions. But $Z_{i+1} \supset Z_{i+1-k}$, so that if

$$Z_{i+1} \cap H_{i-k} = Z_{i+1} \cap H_{i-k-1}$$

then $Z_{i+1} \cap H_{i-k} \subset H_{i-k-1}$, $Z_{i+1-k} \cap H_{i-k} \subset H_{i-k-1}$, and

$$Z_{i+1-k} \cap H_{i-k} \subset Z_{i+1-k} \cap H_{i-k-1},$$

a contradiction with the above strict inclusion. Hence

$$Z_{i+1} \cap H_{i-k} \supset Z_{i+1} \cap H_{i-k-1}$$

with strict inclusion, and the result is established by induction.

We can define for each ordinal α a subgroup H_α of G as follows: H_1 is defined as above. If α is not a limit ordinal, let $\alpha(-)$ be the predecessor of α. If $H_{\alpha(-)}$

is defined, then define H_α by $H_\alpha/H_{\alpha(-)} = H_1\ (G/H_{\alpha(-)})$. If α is a limit ordinal, let

$$H_\alpha = \bigcup_{\beta<\alpha} H_\beta,$$

the set-theoretic union of the H_β. With appropriate but entirely trivial[3] modifications, the prior statements of this paper can be adapted for this extended FC-chain. Similar modifications can be made throughout the remainder of the paper, but these latter are not of such uniform simplicity. Since a detailed discussion at this time of the properties of the extended FC-chain would obscure the central issues, we shall not return to this point in the present work.

7. FC-nilpotency.

LEMMA 4. $$\phi_i^{-1}\ (H_k\ (G/H_i(G)) = H_{i+k}(G) \qquad (k = 0, 1, 2, \ldots).$$

Proof. We use induction on k. For $k = 0$, $\phi_i^{-1}(e^*) = H_i(G)$ (where e^* is the identity of G/H_i), so that the result holds for $k = 0$. $\phi_i^{-1}\ (H_1\ (G/H_i)) = H_{i+1}$, so that the result holds also for $k = 1$. Let us now assume its validity for k. Then $H_{i+k}\ (G)/H_i(G)$ is $H_k\ (G/H_i(G))$. Let Φ_k be the natural mapping on $G/H_i(G)$ onto $G/H_{i+k}(G)$ with kernel $H_k(G/H_i(G)) = H_{i+k}(G)/H_i(G)$.

$$\Phi_k^{-1}(H_1(G/H_{i+k}(G))) = \Phi_k^{-1}\ \phi_{i+k}(H_{i+k+1}),$$

since the case $k = 1$ has been established. But

$$\begin{aligned}\Phi_k^{-1}\ \phi_{i+k}(H_{i+1+k}) &= \Phi_k^{-1}(H_{i+k+1}/H_{i+k})\\ &= \Phi_k^{-1}((H_{i+k+1}/H_i)/(H_{i+k}/H_i)) = H_{i+k+1}/H_i.\end{aligned}$$

However,

$$\begin{aligned}\Phi_k^{-1}(H_1(G/H_{i+k}(G))) &= \Phi_k^{-1}(H_1((G/H_i(G))/(H_{i+k}(G)/H_i(G))))\\ &= \Phi_k^{-1}(H_1((G/H_i(G))/H_k(G/H_i(G))))\\ &= \Phi_k^{-1}(H_{k+1}(G/H_i(G))/H_k(G/H_i(G)))\\ &= H_{k+1}(G/H_i(G)),\end{aligned}$$

and the result is established.

LEMMA 5. *Let Θ be the natural map of G onto G/N where N is a normal subgroup of G. Then $\Theta^{-1}\ H_k(G/N) \supset H_k\ (G)$ $(k = 0, 1, 2, \ldots)$.*

Proof. For $k = 0$, the result is obvious. $H_1(G/N)$ is an FC-group so that $\Theta^{-1}H_1(G/N) \supset H_1(G)$. Now suppose that $R_k = \Theta^{-1}H_k(G/N) \supset H_k(G)$.

$$\begin{aligned}H_1(G/R_k) &\cong H_1((G/N)/(R_k/N)) = H_1((G/N)/H_k(G/N))\\ &\cong H_{k+1}(G/N)/H_k(G/N) \cong (R_{k+1}/N)/(R_k/N)\\ &\cong R_{k+1}/R_k \cong (R_{k+1}/H_k(G))/(R_k/H_k(G)).\end{aligned}$$

But

$$H_1(G/R_k) \cong H_1((G/H_k(G))/(R_k/H_k(G)))$$

[3]See note 1.

so that the latter group is isomorphic to

$$(R_{k+1}/H_k(G))/(R_k/H_k(G)).$$

Hence

$$R_{k+1}/H_k(G) \supset H_1(G/H_k(G)) \cong H_{k+1}(G)/H_k(G),$$

and $R_{k+1} \supset H_{k+1}(G)$, so that the proof is complete.

THEOREM 4. *Let N be a normal subgroup of a group G such that* (1) $N \subset H_n(G)$ *and* (2) *there exists a positive integer k for which G/N is* FC-*nilpotent of* FC-*class k. Then G is* FC-*nilpotent of* FC-*class* $\leqslant n + k$.

Proof. $H_k(G/N) = G/N$.

$$G/H_n(G) \cong (G/N)/(H_n(G)/N);$$

and

$$H_k(G/H_n(G)) = H_{n+k}(G)/H_n(G),$$

by Lemma 4. Hence

$$H_k((G/N)/H_n(G)/N)) \cong (H_{n+k}(G)/N)/(H_n(G)/N).$$

By Lemma 5 (taking G/N for G and $H_n(G)/N$ for N),

$$H_{n+k}(G)/N \supset H_k(G/N) = G/N.$$

Hence $H_{n+k}(G) = G$.

COROLLARY 1. *If $G/Z_n(G)$ is* FC-*nilpotent of* FC-*class k, then $G = H_{n+k}(G)$.*

COROLLARY 2. *If $W_n(G)$ has finite index in G, then $G = H_n(G)$.*

Proof. For $n = 1$, $G/W_1 = G/Z_1$. Since G/Z_1, a finite group, is isomorphic to the group of inner automorphisms [4] of G, there are only a finite number of inner automorphisms of G, and G is an FC-group. For $n > 1$,

$$G/W_n \cong (G/H_{n-1})/(W_n/H_{n-1}).$$

Since $W_n/H_{n-1} = Z_1(G/H_{n-1})$, G/H_{n-1} is an FC-group, by the argument employed for $n = 1$. By the theorem, G is FC-nilpotent of FC-class $\leqslant n$.

COROLLARY 3. *If $G' \subset H_n(G)$ for some non-negative integer n, then G is* FC-*nilpotent of* FC-*class* $\leqslant n + 1$.

Note that if G is FC-nilpotent of FC-class k, then G/N is FC-nilpotent of FC-class $\leqslant k$, where N is a normal subgroup of G. For, by Lemma 5,

$$\theta^{-1}(H_k(G/N)) \supset H_k(G) = G,$$

so that $H_k(G/N) = G/N$. Immediate is

COROLLARY 4. *Let $N \subset H_n(G)$ where N is a normal subgroup of G. Let G be* FC-*nilpotent of* FC-*class t so that G/N is* FC-*nilpotent of* FC-*class k. Then $k \leqslant t \leqslant k + n$.*

8. The FC-chain of a "large" normal subgroup.

THEOREM 5. *Let K be a normal subgroup of finite index in G for which $H_i(G) \subset K (i = 0, 1, 2, \ldots)$. Then $H_i(K) = H_i(G)$ for all such i.*

Proof. Clearly $H_1(K) \supset H_1(G)$. For $x \in H_1(K)$, there exist a finite number of conjugates of x in K. Let the t_i $(i = 1, 2, 3, \ldots, n)$ be the set of representatives of the cosets of K in G. Let g be any element of G. Then there exist $h \in K$ and a positive integer $i < n$ such that $g = ht_i$, whence $g^{-1}xg = t_i^{-1}(h^{-1}xh)t_i$. There are only a finite number of possibilities for the $h^{-1}xh$ since $x \in H_1(K)$ and $h \in K$. Hence there are only a finite number of $g^{-1}xg$ for fixed $x \in H_1(K)$. Thus $x \in H_1(G)$, and $H_1(K) = H_1(G)$.

Now suppose that $H_i(K) = H_i(G)$. Since G/K is a finite group, the index of $K/H_i(G)$ in $G/H_i(G)$ is finite. Since $K \supset H_{i+1}(G)$,

$$K/H_i(G) \supset H_{i+1}(G)/H_i(G), \qquad H_1(K/H_i(G)) = H_1(G/H_i(G))$$

by the above argument on H_1. Then

$$H_{i+1}(G) = \phi_i^{-1}(H_1(K/H_i(G))) = \phi_i^{-1}(H_1(K/H_i(K))),$$

since $H_i(K) = H_i(G)$. Since $H_{i+1}(G) \subset K$, it follows that $H_{i+1}(K) = H_{i+1}(G)$.

COROLLARY. *Let G be an extension of an* FC*-nilpotent group K by a finite, non-trivial group F. Then there exists a positive integer i for which $H_i(G) \not\subset K$.*

Proof. If each $H_j(G) \subset K$ $(j = 1, 2, 3, \ldots)$ then, by the theorem, each $H_j(G) = H_j(K)$. In particular, $H_n(G) = H_n(K) = K$, where n is the FC-class of K. But then

$$H_1(G/K) = H_1(G/H_n(G)) = H_{n+1}(G)/H_n(G).$$

Since $H_1(G/K) = F$ is a non-trivial group, $H_{n+1}(G) \neq H_n(G) = K$. But $H_{n+1}(G) = H_{n+1}(K)$, by the theorem, and $H_{n+1}(K) = H_n(K) = K$, a contradiction.

9. Groups for which H_1 is trivial.

THEOREM 6. *Let $H_n(G)$ be a direct summand of the group G. Then*

$$H_{n+k}(G) = H_n(G) \qquad (k = 1, 2, 3, \ldots).$$

Proof. $G = H_n(G) \oplus K$, where $K \cong G/H_n(G)$. Hence $H_1(K) \cong H_{n+1}(G)/H_n(G)$. Consider ordered pairs (e, x), where e is the identity of $H_n(G)$ and $x \in H_1(K)$. It follows that $(e, x) \in H_1(G)$. Hence $(e, x) \in H_n(G)$, so that $x = e'$, the identity of K. Thus $H_1(K) = (e')$ and $H_{n+1}(G) = H_n(G)$. The result follows at once.

If the FC-chain breaks off before or at $H_n(G)$, then $H_1(G/H_n(G))$ is the trivial group, and conversely. Thus $G/H_n(G)$ has no non-trivial H_1-group and has, as a consequence, no non-trivial centre and is isomorphic to the group of its inner automorphisms.

For an automorphism α of G, let $F(\alpha)$ denote the set of all points which are fixed under α. This set of fixed points is, as is well known, a subgroup of G. Let $J(G)$ be the group of inner automorphisms of G, and let $A(G)$ be the group of automorphisms of G. Recall the definition of mutual truncation in §2. We then have

THEOREM 7. *Let G be a group with mutual truncation at index $< n$. Then $J(G)$ has mutual truncation at index $n - 1$ (if $n > 1$).*

Proof. For any index k, $H_k(G/H_n(G))$ is trivial, by Lemma 4. Since $H_n(G) = Z_n(G)$,

$$H_k(G/H_n(G)) \cong H_k(J(G)/Z_{n-1}(J(G))).$$

By Lemma 5, $Z_{n-1}(J(G)) \supset H_k(J(G))$. Take $k = n$ for the result.

COROLLARY 1.[4] *If G has mutual truncation at index < 1, and if U is any group extension of $J(G)$, then each $J(G) \cap H_n(U)$ is trivial.*

Proof. By the theorem, $J(G)$ has mutual truncation at index 0. If S and T are groups with $S \subset T$, it is easy to see that $S \cap H_n(T) \subset H_n(S)$ for every n. Take $S = J(G)$ and $T = U$ for the result.

We should note[4] that the condition $H_1(G) \cap G' = (e)$ implies mutual truncation for G at index < 1. For, if N is a normal subgroup of G, then $G' \cap N = (e)$ implies that $gxg^{-1}x^{-1} = e$ for every $g \in G$ and for every $x \in N$. Hence $N \subset Z_1(G)$, and $H_1(G) = Z_1(G)$. Then $Z_1(G) \cap G'$ is trivial. But the latter has one of two consequences: (1) $Z_1(G) = (e)$, whence we have mutual truncation at index 0, or (2) $G' = (e)$, whence G is abelian, so that we have mutual truncation at index < 1.

COROLLARY 2. (a) *Let G have mutual truncation at index < 1, and let $J(G)$ be* FC-*nilpotent. Then G is abelian.* (b) *If $H_1(G)$ is trivial, then $A(G)$ is* FC-*nilpotent if and only if G is trivial.*

Proof. (a) By Corollary 1, each $H_n(J(G))$ is trivial. Since $J(G)$ is FC-nilpotent, $J(G)$ must be trivial so that G is abelian. (b) If $A(G)$ is FC-nilpotent, its subgroup $J(G)$ is also FC-nilpotent. $H_1(G) = (e)$ implies that $Z_1(G) = (e)$. By (a), G is abelian, so that $G = H_1(G) = (e)$.

Part (b) of the above corollary shows that if G is "badly" non-abelian and infinite, then its automorphism group cannot be finite, abelian, nilpotent, FC or, in general, FC-nilpotent.

COROLLARY 3. *If the* FC-*chain of G breaks off after a finite number of steps (say, at index n), then G is* FC-*nilpotent if and only if $A(G/H_n(G))$ is* FC-*nilpotent.*

[4]See note 1.

Proof. $H_1(G/H_n(G))$ is trivial, since $H_{n+1}(G) = H_n(G)$. By Corollary 2(b), $G/H_n(G)$ is trivial, and $G = H_n(G)$.

THEOREM 8. *Let G be a group with mutual truncation at index $< r$. Let U be any group extension of $J(G)$ with the property $J(G) \subset U \subset A(G)$, where the inclusions need not be strict. Let α be in $H_j(U)$, and let α_r be the automorphism induced on $G/Z_r(G)$ by α. Then*

$$D(F(\alpha_r);\quad G/Z_r(G)) = G/Z_r(G).$$

Proof. If $j = 0$ or if G is trivial, the proof is immediate. If G is non-trivial and if $j > 1$, consider a fixed $g \in G, g \neq e.$ $\alpha \in H_j(U)$ then implies the existence of integers $m < n$ and of $\phi \in H_{j-1}(U)$ such that

$$9.1 \qquad g^{-m}\,\alpha(g^m\,xg^{-m})\,g^m = g^{-n}\,\alpha(g^n\,\phi(x)g^{-n})\,g^n$$

for every $x \in G$. If we write $\alpha(u) = g$ and $k = g^{-m}\,u^{m-n}\,g^n$, then $\phi(x) = k^{-1}xk$ for every $x \in G$. Thus

$$\phi \in J(G) \cap H_{j-1}(U) \subset H_{j-1}(J(G)).$$

But the latter group is included in $Z_{r-1}(J(G))$, by the proof of Theorem 7. Thus $k \in Z_r(G)$. Now $k^{-1}xk = xv(x)$ where $v(x) \in Z_{r-1}(G)$ if $r > 1$ and $v(x) = e$ if $r = 0$. If we write $\alpha(g) = h$, 9.1 can be simplified to

$$9.2 \qquad g^{n-m}\,h^m\,\alpha(x)\,h^{-m} = h^n\,\alpha(x)\,\alpha(v)\,h^{-n}\,g^{n-m},$$

or

$$9.3 \qquad h^{-n}\,g^{n-m}\,h^m\,\alpha(x) \equiv \alpha(x)\,h^{-n}\,g^{n-m}\,h^m \qquad \text{mod } Z_{r-1}(G).$$

Since α is an automorphism, $\alpha(x)$ ranges over all of G, and $h^{-n}\,g^{n-m}\,h^m \in Z_r(G)$. Thus $g^{n-m} \equiv h^{n-m}$ mod $Z_r(G)$ for every $g \in G$ (where we understand that m and n are functions of g and α). Remembering that $h = \alpha(g)$, we see that the conclusion of the theorem follows at once.

COROLLARY 1. *If G has mutual truncation at index $< r$, if $Z_r(G)$ is a periodic group, and if $\alpha \in H_j(U)$ where $J(G) \subset U \subset A(G)$, then $D(F(\alpha); G) = G$.*

COROLLARY 2. (a) *Let G be a group for which $Z_1(G)$ is trivial. For $\alpha \in H_1(U)$ where $J(G) \subset U \subset A(G)$, $D(F(\alpha); G) = G$.* (b) *Let G be a finite group for which $Z_1(G)$ is trivial. For $\alpha \in A(G)$, $D(F(\alpha); G) = G$.*

Proof. (a) In the proof of the theorem we can take $\phi = I$, the identity automorphism. Then 9.1 in the proof reduces to

$$h^{-n}\,g^{n-m}\,h^m\,y = yh^{-n}\,g^{n-m}\,h^m$$

where $\alpha(x) = y$. Since α is an automorphism and since $Z_1(G) = (e)$, $h^{-n}\,g^{n-m}\,h^m = e$, and (a) follows directly. (b) is a trivial consequence of (a).

THEOREM 9. *Let G be a group with mutual truncation at index $< r$. Let U be any group extension of $J(G)$ with the property $J(G) \subset U \subset A(G)$, where the*

inclusions need not be strict. If $\alpha \in H_j(U)$, *then* $F(\alpha_r)$ *has a finite index in* $G/Z_r(G)$.

Proof. There exists a finite (but not necessarily unique) set of elements $\{g_i\}$ $(i = 1, 2, \ldots, N)$ in G such that to $g \in G$, there exists an index i and a mapping $\phi \in H_{j-1}(U)$ with

$$g^{-1}\alpha(gxg^{-1})\, g = g_i^{-1}\alpha(g_i\,\phi(x)\, g_i^{-1})\, g_i$$

for every $x \in G$. As in the proof of Theorem 8, $\phi(x) = xv(x)$, where $v(x) \in Z_{r-1}(G)$ if $r > 1$, and $v(x) = e$ if $r = 0$. It follows that

$$g_i^{-1}\alpha^{-1}(g_i)\,\alpha^{-1}(g^{-1})\, g \in Z_r(G)$$

or that $\alpha(gg_i^{-1}) \equiv gg_i^{-1} \bmod Z_r(G)$. The theorem follows at once. A trivial rearrangement of the last step shows that

$$g^{-1}\alpha(g) \equiv g_i^{-1}\alpha(g_i) \qquad \bmod Z_r(G),$$

as we should expect in light of [**3**, p. 165, (c′)].

COROLLARY 1.[5] *If* $H_1(G)$ *is trivial and if* $\alpha \in H_j(U)$, *where* $J(G) \subset U \subset A(G)$, *then* $F(\alpha)$ *has finite index in* G.

COROLLARY 2. *Let* G *be a group for which* $Z_1(G)$ *is trivial. If* $\alpha \in H_1(U)$, *where* $J(G) \subset U \subset A(G)$, *then* $F(\alpha)$ *has finite index in* G.

Proof. In the proof of the theorem we can take ϕ to be the identity map. The rest of the argument follows without difficulty.

Following common custom, a group will be called *complete* if for each positive integer n, the set of all x^n $(x \in G)$, is a set of generators for G. By $T_n(G)$, where n is a fixed positive integer, we shall mean the set of all $\alpha \in A(G)$ for which $\alpha(x) \equiv x \bmod Z_n(G)$ for every $x \in G$. If $n = 1$, we have the so-called *normal* or *central* automorphisms [**6**]. $T_0(G)$ is to consist of the identity automorphism of G, alone.

THEOREM 10. *Let* G *be a complete group which has mutual truncation at index* r. *Let* U *be an extension of* $J(G)$ *with the property* $J(G) \subset U \subset A(G)$, *where the inclusions need not be strict. Then* $H_j(U) \subset T_r(G)$ $(j = 1, 2, \ldots)$.

Proof. Suppose $\alpha \in H_j(U)$. To each $x \in G$, there exists, by the proof of Theorem 8, a positive integer $t(x)$ such that

$$\alpha(x^{t(x)}) \equiv x^{t(x)} \qquad \bmod Z_r(G).$$

Moreover, there exists a uniform bound $M = M(\alpha) > t(x)$ for all $x \in G$, since $\alpha \in H_j(U)$. Let $N = M!$. Then $\alpha(x^N) \equiv x^N \bmod Z_r(G)$. Since the set of all x^N is a set of generators of G, $\alpha(g) \equiv g \bmod Z_r(G)$ for every $g \in G$, and $\alpha \in T_r(G)$.

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[5] See note 1.

COROLLARY 1. *If $H_1(G)$ is trivial for a complete group G, then $H_1(U)$ is trivial, where $J(G) \subset U \subset A(G)$.*

Proof. Note that the index r of truncation is 0. Alternately, we can prove a stronger result:

COROLLARY 2. *If $Z_1(G)$ is trivial for a complete group G, then $H_1(U)$ is trivial, where $J(G) \subset U \subset A(G)$.*

Proof. Using the proof of Corollary 2 to Theorem 8, we can modify the proof of the present theorem to show that $\alpha(x^N) = x^N$ for every x.

10. Examples of FC-nilpotent groups. Consider two countable classes of copies of I_2, the group of integers modulo 2. Let the generators of these groups be denoted by the e_i ($i = 1, 2, 3, \ldots$) and the f_j ($j = 1, 2, 3, \ldots$). Form the free product F of all the members of these two classes of copies of I_2. Impose the relations (1) $e_i x = x e_i$, for every generator e_i of the first class and for every $x \in F$; and (2) $f_i f_j = f_j f_i e_i e_j$ for all i and j. Call the resulting group G. Then every word in G can be given the unique canonical form $f_{i_1} f_{i_2} \ldots f_{i_n} E$ where E is a word in the e_i's and $i_1 < i_2 < \ldots < i_n$ if the class of the i_j's is non-void. It is easy to prove that $Z_1(G)$ is the set of all elements generated by the e's alone. If the number of f's in the canonical form of a word is even, then the word has no more than two conjugates in G, while if the number is odd, there are an infinite number of conjugates of the word in G. All the words of even "f-length" form the subgroup $H_1(G)$, and this subgroup is distinct from both $Z_1(G)$ and G. It is clear that $x^2 \in Z_1(G)$ for every $x \in G$, so that $G = D(Z_1(G); G)$. It is not difficult to show that $G/H_1(G) \cong I_2$, so that $G = H_2(G)$. We thus have an example of an FC-nilpotent group of FC-class 2.

The referee has pointed out the following: Let G be a free group on two or more generators. We construct the *lower central series* [6] of G as follows: $G(0) = G$; $G(1) = (G, G)$; $G(i + 1) = (G, G(i))$. Then $G/G(c)$ is an example of an FC-nilpotent group of FC-class c for every positive integer c.

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DIFFERENTIABLE POINTS IN THE CONFORMAL PLANE

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1. Introduction. The purpose of this note is the classification of the differentiable points on curves in the conformal plane. We introduce tangent and osculating circles at such points and study the intersection and support properties of these circles.

Our paper is related to the case $n = 3$ of the classification of the differentiable points on curves in projective n-space given in [**4**]. The connecting link is a stereographic projection of a curve in the conformal plane on a spherical one. Naturally, this note is also connected with the many discussions of the curvature and the osculating circles of curves in the Euclidean plane [**1**; **2**; **3**; **5**].

2. Pencils of circles. In the following, $P, Q, \ldots$ denote points in the conformal plane; $C, C', \ldots$ denote oriented circles. Such a circle C decomposes the plane into two open regions, its *interior* $\underline{C}$ and its *exterior*[1] $\bar{C}$. The circle through three mutually distinct points, P, Q, and R will occasionally be denoted by $C(P, Q, R)$.

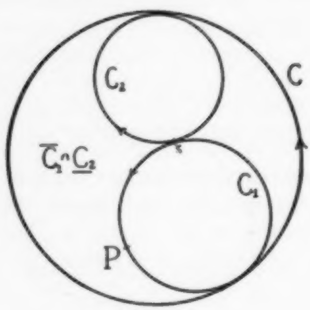

FIG. 1

The set of all circles that intersect two given circles at right angles form a linear pencil π of circles. A pencil π of the *first* kind possesses two fundamental points such that π is identical with the set of all circles through these points. A pencil π of the *second* kind has one fundamental point and is identical with the set of those circles that touch a given circle at that point. If π is of the *third* kind, then any two circles of π are disjoint. To any pencil π and to any

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[1] The meaning of these terms is not identical with that given in ordinary plane geometry. The interior of the oriented circle C lies at its left; cf. Fig. 1.

point Q which is not a fundamental point of π, there exists one and only one circle $C(\pi, Q)$ of π through Q. We consider the fundamental point of a pencil π of the second kind as a point-circle belonging to π.

The set of the circles perpendicular to a given pencil π form again a pencil ρ. If π is of the pth kind, then ρ is of the $(4 - p)$th kind ($p = 1, 2, 3$). The relation between π and ρ is involutory. If $C \subset \rho$, then every circle of π meets C, and C contains a fundamental point of π if and only if π is of the second kind.

3. Convergence. We call the sequence of points $P_1, P_2, \ldots$ *convergent* to P if there exists to every C with $P \subset \underline{C}$ a number $n = n(C)$ such that $P_\nu \subset \underline{C}$ if $\nu > n$.

In the same way, the convergence of circles to a point is defined.

We call the sequence $C_1, C_2, \ldots$ convergent to C if there exists to every pair $C' \subset \underline{C}$ and $C'' \subset \bar{C}$ a number $n = n(C', C'')$ such that $C' \subset \underline{C}_\nu$ and $C'' \subset \bar{C}_\nu$ for every $\nu > n$.

4. Support and intersection at a point of an arc. An *arc* A is the continuous image of a closed interval. Thus if a sequence of points of that *parameter interval* converges to a point s, then their image points converge to the image of s. We shall use the same letters $s, s', \ldots$ to denote both the *parameter* (i.e., the points of the parameter interval) and their images on A. The *end* (*interior*) points of A are the images of the end (interior) points of the parameter interval.

A *neighbourhood* of s on A is the image of a neighbourhood of the parameter s on the parameter interval. If s is an interior point of A, this neighbourhood is decomposed by s into two (open) *one-sided* neighbourhoods.

From our definition, different points of A, i.e., points with different parameters, may coincide with the same point of the conformal plane. However, we shall assume that each point s of A has a neighbourhood such that no other point of that neighbourhood coincides with s. (The notation $P \neq s$ will indicate that the points P and s do not coincide.)

Suppose s is an interior point of A. Then we call s a *point of support* (*intersection*) with respect to the circle C if a sufficiently small neighbourhood of s is decomposed by s into two one-sided neighbourhoods which lie in the same region (in different regions) bounded by C. C is then called a *supporting* (*intersecting*) circle of A at s. Thus C supports A at s if $s \not\subset C$. By definition, the point-circle s always supports A at s.

It can happen that every neighbourhood of s has points $\neq s$ in common with C. Then C neither supports nor intersects A at s.

5. Tangent circles of s. Suppose the point $s \subset A$ satisfies the following

CONDITION I. To every point $P \neq s$ and to every sequence of points $s' \to s$ ($s' \neq s$; $s' \subset A$) there exists a circle C' such that $C(s', s, P) \to C'$.

Obviously, the *tangent circle* C' is independent of the choice of the sequence s'. We admit the point s itself as the tangent point-circle of A at s.

THEOREM 1. *The set $\tau = \tau(s)$ of all the tangent circles of A at s is a pencil of the second kind with the fundamental point s.*

Proof. Let P, Q, R be three mutually distinct points. If the point $R' \neq R$ converges to R, then the angle between the circles $C(R', R, P)$ and $C(R', R, Q)$ converges to zero.[2] We choose $R = s$ and $R' = s'$. Since the angle between two circles depends on them continuously, we conclude that any two tangent circles at s touch each other at that point. Thus two tangent circles that have another point in common are identical. In particular, there exists one and only one tangent circle at s through each point different from s.

Suppose the circle C touches the tangent circle C' at s. Let $P \subset C$, $P \neq s$. Then C also touches the tangent circle through P. Hence C is equal to that circle.

By Thereom 1, the tangent circle $C(\tau, P)$ through P depends continuously on P as long as $P \neq s$.

THEOREM 2. *Suppose the point $s \subset A$ satisfies Condition* I. *Let π be a pencil of the second kind with s as its fundamental point; $\pi \neq \tau$. If the points s' converge to s $(s' \neq s)$, then $C(\pi, s') \to s$.*

Proof. If our statement were false, there would exist a circle C such that $s \subset \underline{C}$ and a sequence of points $s' \to s$ $(s' \neq s)$ such that $C(\pi, s') \not\subset \underline{C}$ for each s'. Let C_1 and C_2 be the two circles of π that touch C. We may assume that π is oriented such that C lies in the closure of $\bar{C}_1 \cap \underline{C}_2$. Then this closed domain also contains the circles $C(\pi, s')$ and therefore the points s' (cf. Fig. 1).

Let P be any point of C_1; $P \neq s$. If a sequence of points Q converges to s through the above domain, then the circles $C(s, P, Q)$ converge to C_1. Choosing $Q = s'$, we obtain $C_1 = C(\tau, P)$, while $\tau \neq \pi$. This is a contradiction.

6. Non-tangent circles. Let s be an interior point of A. Suppose again that s satisfies Condition I (cf. §5).

THEOREM 3. *Every non-tangent circle either supports or intersects A at s.*

Proof. If the circle C neither supports nor intersects A at s, then $s \subset C$ and there exists a sequence of points $s' \to s$ such that $s' \subset A \cap C$ and $s' \neq s$. Let $P \subset C$, $P \neq s$. Then $C = C(s', s, P)$ for each s', and Condition I implies $C = C(\tau, P)$.

THEOREM 4. *Non-tangent circles through s all intersect or all support.*

Proof (cf. Fig. 2). Let C_1 and C_2 be two distinct non-tangent circles through s. We assume at first that they have another point $P \neq s$ in common. Suppose for example, that C_1 intersects and C_2 supports at s. Thus $A \cap \underline{C}_1$ and $A \cap \bar{C}_1$ are non-void. Without restriction of generality, we may assume that $A \subset \bar{C}_2$.

If $s' \subset A \cap \underline{C}_1$, then $C(s, s', P)$ lies in the closure of $(\underline{C}_1 \cap \bar{C}_2) \cup (\bar{C}_1 \cap \underline{C}_2)$.

[2] The circles themselves need not be convergent.

By having s' converge to s, we conclude that $C(\tau, P)$ lies in the same closed domain. By having s' converge to s through $A \cap \bar{C}_1$, we obtain symmetrically

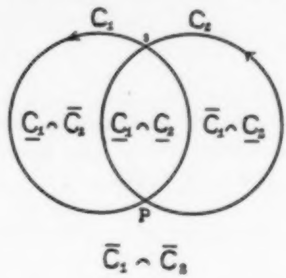

FIG. 2

that $C(\tau, P)$ also lies in the closure of $(\underline{C}_1 \cap \underline{C}_2) \cup (\bar{C}_1 \cup \bar{C}_2)$. Hence $C(\tau, P)$ lies in the intersection $C_1 \cup C_2$ of these two domains, i.e., $C(\tau, P)$ is either C_1 or C_2, contrary to our assumptions. Thus C_1 and C_2 either both support or both intersect.

If C_1 and C_2 meet only at s, then they touch at that point. Choose any non-tangent circle C_3 through s that does not belong to the pencil through C_1 and C_2. From the above, C_1 and C_3, and also C_3 and C_2, either both support or both intersect. Hence our statement remains valid for C_1 and C_2 also in this case. This completes our proof.

7. Differentiable points. We call the point s *differentiable* if it satisfies not only Condition I but also the following one:

CONDITION II. If s' converges to s $(s' \neq s)$, then $C(\tau, s')$ converges to some circle $C(s)$ (cf. §5).

If s is differentiable, then the *osculating circle* $C(s)$ is obviously independent of the choice of the points s'. Furthermore, τ being closed, we certainly have $C(s) \subset \tau$.

THEOREM 5. *The point s is differentiable if and only if the limit circle*

$$\lim_{s' \to s} C(\pi, s') \qquad s' \neq s$$

exists for every π.

Proof. This follows at once when the continuity of A at s and the Conditions I and II are combined with §5.

As a consequence of Theorem 5, we may define

$$C(\pi, s) = \lim_{s' \to s} C(\pi, s') \qquad s' \neq s$$

if s is a fundamental point of π. In this way $C(\pi, s')$ will become continuous at s for every π.

The following example shows that Condition II does not follow from Condition I. We introduce retangular cartesian coordinates x, y. Then the arc A defined by

$$x = s,\ y = \begin{cases} (1 - \sqrt{1 - s^2}) \sin s^{-1}, & 0 < |s| < 1 \\ 0, & s = 0 \end{cases}$$

lies between the two circles $x^2 + (y \pm 1)^2 = 1$. In particular A satisfies Condition I at $s = 0$, and τ consists of all the circles that touch the x-axis at the origin. Since every neighbourhood of $s = 0$ on A has points in common with both of the above circles, II does not hold.

8. Non-osculating tangent circles of s. Let s be a differentiable interior point of A.

THEOREM 6. *Every non-osculating circle either supports or intersects at* s.

Proof. If the circle C neither supports nor intersects at s, then $C \subset \tau$ (cf. §6) and there exists a sequence of points $s' \to s$, $s' \neq s$ on C. Thus $C = C(\tau, s')$ for each s'. From Condition II,

$$C = \lim_{s' \to s} C(\tau, s') = C(s).$$

THEOREM 7. *If* $C(s) \neq s$, *then every non-osculating tangent circle supports.*

Proof. Suppose that $C \subset \tau$. $C \neq C(s)$. If a sequence of points s' exists such that $s' \subset A \cap \bar{C}$, $s' \neq s$, $s' \to s$, then each $C(\tau, s')$ lies in the closure of $\bar{C}$. Hence $C(s)$ will lie in the same domain and therefore even in $s \cup \bar{C}$. Similarly the existence of a sequence $s' \subset \underline{C}$, $s' \neq s$, $s' \to s$ implies $C(s) \subset s \cup \underline{C}$. Since $(s \cup \bar{C}) \cap (s \cup \underline{C}) = s$, $C(s) = s$.

It remains to consider the case that $C(s)$ is the point-circle s. In this case we prove

THEOREM 8. *Either the other tangent circles all support or they all intersect.*

Proof. Let C_1 and C_2 be two distinct non-osculating tangent circles. We may assume that τ is oriented such that $C_2 \subset (s \cup \bar{C}_1)$; thus $C_1 \subset (s \cup C_2)$ (compare

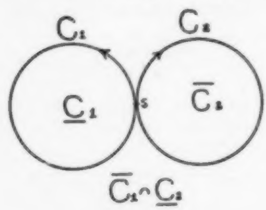

FIG. 3

Fig. 3). From the above, there exists a neighbourhood N of s on A which has no other points in common with $C_1 \cup C_2$.

<table>
<tr><th>Charac-teristic</th><th>Non-tangent circles through s</th><th>Tangent circles $\neq C(s)$</th><th colspan="2">$C(s)$</th><th colspan="3">Examples</th></tr>
<tr><td>(1, 1, 1)</td><td>intersect</td><td rowspan="4">support</td><td rowspan="4">$C(s) \neq s$</td><td>intersects</td><td rowspan="4">$n < m$</td><td>$n = 1$, $m = 0$</td><td>regular point</td></tr>
<tr><td>(1, 1, 2)</td><td>intersect</td><td>supports</td><td>$n = m = 1$</td><td>vertex</td></tr>
<tr><td>(2, 2, 1)</td><td>support</td><td>intersects</td><td>$n = 0$, $m = 1$</td><td>cusp of the first kind[3]</td></tr>
<tr><td>(2, 2, 2)</td><td>support</td><td>supports</td><td>$n = m = 0$</td><td>cusp of the second kind[3]</td></tr>
<tr><td>$(1, 1, 2)_0$</td><td>intersect</td><td>support</td><td colspan="2" rowspan="4">point-circle</td><td rowspan="4">$n > m$</td><td>$n = m = 1$</td><td rowspan="4"></td></tr>
<tr><td>$(1, 2, 1)_0$</td><td>intersect</td><td>intersect</td><td>$n = 1$, $m = 0$</td></tr>
<tr><td>$(2, 1, 1)_0$</td><td>support</td><td>intersect</td><td>$n = 0$, $m = 1$</td></tr>
<tr><td>$(2, 2, 2)_0$</td><td>support</td><td>support</td><td>$n = m = 0$</td></tr>
<tr><td>$(1, 1, \infty)$</td><td>intersect</td><td rowspan="2">support</td><td colspan="2" rowspan="2">neither intersects nor supports</td><td rowspan="2">$n < m$</td><td>$n = 1$</td><td rowspan="2"></td></tr>
<tr><td>$(2, 2, \infty)$</td><td>support</td><td>$n = 0$</td></tr>
</table>

[3]These names apply if we choose the infinite point on $C(s)$ but different from s.

Suppose that C_1, for instance, supports, while C_2 intersects A at s. Then some points of N lie in $\bar{C}_2$ and therefore in $\bar{C}_1$. Hence $N \subset s \cup \bar{C}_1$. Furthermore, from our assumption, there is a sequence of points s' which converge to s through C_2 and therefore through $\bar{C}_1 \cap \underline{C}_2$. Consequently $C(\tau, s') \subset s \cup (\bar{C}_1 \cap \underline{C}_2)$. Hence $C(\tau, s')$ cannot converge to $C(s) = s$ if s' converges to s. This is a contradiction.

9. A classification of the differentiable points. Sections 6 and 8 yield a classification of the differentiable interior points of A, as on page 517. The first eight examples refer to the curves $x = s^n$, $y = s^{n+m}$; the last two refer to $x = s^n$, $y = s^{n+m} \sin s^{-1}$. In all these cases we consider the point $s = 0$. Congruences are mod 2.

The *characteristic* (a_0, a_1, a_2), where $a_0, a_1 = 1$ or 2, and $a_2 = 1, 2$, or ∞, has the following properties: a_0 is even or odd according as the non-tangent circles through s support or intersect; $a_0 + a_1$ is even or odd according as the non-osculating tangent circles support or intersect; $a_0 + a_1 + a_2$ is even if $C(s)$ supports, odd if $C(s)$ intersects, while $a_2 = \infty$ if $C(s)$ neither supports nor intersects. We shall use the notation $(a_0, a_1, a_2)_0$ whenever $C(s)$ is the point-circle s (cf. §4).

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VARIATION OF CONGRUENCES OF CURVES OF AN ORTHOGONAL ENNUPLE IN A RIEMANNIAN SPACE

T. K. PAN

1. Introduction. Consider any three congruences of an orthogonal ennuple at a point of a Riemannian space. When one congruence is moved by local and a second congruence is moved by parallel displacement in the direction of the third congruence, the rate of change of cosine of the angle between the first two congruences is well known as Ricci's coefficient of rotation and has been extensively studied. It is the purpose of this note to investigate the corresponding rate of change, when the third congruence is replaced by an arbitrary one, in connection with parallelism and equidistance of congruences as studied by Miss Peters [**2**; **3**].

The notation of Eisenhart [**1**] will be used for the most part.

2. Definition. A congruence of curves is uniquely determined by a vector field, which at a point is tangent to the curve of the congruence through the point. Let $\lambda_{h|}$ $(h = 1, \ldots, n)$, be the unit tangents to n congruences of an orthogonal ennuple in a Riemannian space V_n, whose first fundamental form $g_{ij}\,dx^i\,dx^j$ is positive definite. We assume that the components $\lambda_{h|}^{\,i}$ and the coefficients g_{ij} are real analytic functions of the coordinates x's.

Let

$$u^i = \sum_{m=1}^{n} c_m \lambda_{m|}^{\,i}$$

be an arbitrary but fixed congruence of curves C, where $c_h = u^i \lambda_{h|i} = \cos\theta_h$, θ_h being the angle which C makes with the congruence $\lambda_{h|}$. The unit tangent vector to C is given by

$$\xi^i = \sum_{m=1}^{n} c_m \lambda_{m|}^{\,i} \Big/ \left(\sum_{m=1}^{n} c_m^{\,2}\right)^{1/2}. \tag{2.1}$$

When $\lambda_{h|}$ is displaced locally and $\lambda_{k|}$ parallelly along C, the rate at which the cosine of the angle between them changes is measured by p_{hk}, where

$$p_{hk} = \lambda_{h|i,j}\,\lambda_{k|}^{\,i}\xi^j = \sum_{m=1}^{n} c_m \gamma_{hkm} \Big/ \left(\sum_{m=1}^{n} c_m^{\,2}\right)^{1/2}, \tag{2.2}$$

γ_{hkm} being Ricci's coefficients of rotation of the orthogonal ennuple.

The vanishing of the partial derivatives of p_{hk} with respect to the c's requires

$$c_1 : c_2 : \ldots : c_n = \gamma_{hk1} : \gamma_{hk2} : \ldots : \gamma_{hkn}. \tag{2.3}$$

Substituting (2.3) into (2.2) we find that the extreme of p_{hk} as ξ^i varies is equal to

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$$(2.4)\qquad \gamma_{hk} = \left(\sum_{m=1}^{n} \gamma_{hkm}{}^2\right)^{\frac{1}{2}}.$$

The preceding result (2.4) is obtained with the assumption that p_{hk} is a function of the c's. Suppose p_{hk} is independent of the c's. Then from (2.2) we have

$$(2.5)\qquad \gamma_{hkm} = 0 \qquad (m = 1, \ldots, n),$$

and consequently $p_{hk} = 0$. If zero is considered as the extreme of zero's, this extreme value can be obtained also by the substitution of (2.5) into (2.4). Hence the formula (2.4) is valid whether p_{hk} is a function of the c's or not.

We call γ_{hk} *the variation of* $\lambda_{h|}$ *with respect to* $\lambda_{k|}$. When m in (2.1) does not take the value k, we call γ_{hk} *the subvariation of* $\lambda_{h|}$ *with respect to* $\lambda_{k|}$.

3. Properties. Since $\gamma_{hkm} + \gamma_{khm} = 0$ and

$$g_{ij}\left\{\sum_{m=1}^{n} \gamma_{hkm}\,\lambda_{m|}{}^{i} \Big/ \left(\sum_{m=1}^{n} \gamma_{hkm}{}^2\right)^{\frac{1}{2}}\right\}\left\{\sum_{m=1}^{n} \gamma_{khm}\lambda_{m|}{}^{j} \Big/ \left(\sum_{m=1}^{n} \gamma_{khm}{}^2\right)^{\frac{1}{2}}\right\} = -1,$$

it is evident that *the variation of any two orthogonal congruences with respect to each other are equal and the corresponding directions of displacement are coincident but opposite in sense.*

The variation of any congruence with respect to itself is zero, since $\gamma_{hhm} = 0$ for $m = 1, \ldots, n$. The variation of a congruence with respect to another congruence of an ennuple, that is γ_{hk}, $h \neq k$, is zero if and only if the rate of change p_{hk} is zero along every congruence of the ennuple and hence along any congruence of curves in the V_n. In both cases the curve of displacement is arbitrary.

All the congruences of an orthogonal ennuple are normal if and only if

$$(3.1)\qquad \gamma_{hkm} = 0 \qquad (h, k, m = 1, \ldots, n;\ h \neq k \neq m \neq h).$$

To such an ennuple corresponds an n-ply orthogonal system of hypersurfaces. Substitution of (3.1) into (2.4) gives

$$(3.2)\qquad \gamma_{hk} = (\gamma_{hkk}{}^2 + \gamma_{khh}{}^2)^{\frac{1}{2}} = (1/r_{hk}{}^2 + 1/r_{kh}{}^2)^{\frac{1}{2}} = (1/b_{hk}{}^2 + 1/b_{kh}{}^2)^{\frac{1}{2}},$$

where $1/r_{hk}$ and $1/b_{hk}$ are respectively [**2**, pp. 108–109] the angular and distantial spreads of the congruence $\lambda_{h|}$ with respect to the congruence $\lambda_{k|}$. Hence we have

THEOREM 3.1. *In an orthogonal ennuple of normal congruences, a congruence has zero variation with respect to a second if and only if each congruence is equidistant with respect to the other or each congruence is parallel with respect to the other.*

When the variation (subvariation) of a congruence of an orthogonal ennuple with respect to another congruence of the ennuple is numerically equal to a coefficient of rotation, we call the former congruence a *principal* (*subprincipal*)

congruence with respect to the latter congruence. Hence we have from (3.2).

THEOREM 3.2. *A congruence of an orthogonal ennuple of normal congruences is a principal congruence with respect to another congruence of the ennuple if and only if one is parallel along the other.*

Let $n-1$ mutually orthogonal congruences $\lambda_\alpha|$ ($\alpha = 1, \ldots, n-1$), orthogonal to a normal congruence $\lambda_n|$, be canonical with respect to the latter. Necessary and sufficient conditions for this [**1**, p. 128] are $\gamma_{n\alpha\beta} = 0$, $\alpha \neq \beta$. Such a normal congruence is normal to the hypersurface V_{n-1} determined by the $\lambda_\alpha|$'s, which are the lines of curvature of the V_{n-1}.

The subvariation $\gamma_{\alpha n}$ of $\lambda_\alpha|$ with respect to $\lambda_n|$ is then equal to $|\gamma_{\alpha n\alpha}|$. Hence we have

THEOREM 3.3. *A congruence of an orthogonal ennuple is a subprincipal congruence with respect to a congruence of the ennuple if the former is any one of the $n-1$ principal directions in the hypersurface normal to the latter.*

The difference between the squares of the variation and the subvariation of $\lambda_h|$ with respect to $\lambda_n|$ is found from (2.4) to be

$$\sum_{m=1}^{n} \gamma_{hnm}^{\;2} - \sum_{\alpha=1}^{n-1} \gamma_{hn\alpha}^{\;2} = \gamma_{hnn}^{\;2}.$$

Let μ^i denote the angular spread vector of $\lambda_h|$ with respect to $\lambda_n|$, that is $\mu^i = \lambda_h|_{i,j}\lambda_n|^j$. The projection of the vector μ^i in the direction $\lambda_n|$ is called the tendency of $\lambda_h|$ in that direction. Its value is

$$\mu^i \lambda_n|_i = \lambda_h|_{i,j}\lambda_n|^i \lambda_n|^j = \gamma_{hnn}, \tag{3.3}$$

which is equal to zero for $h = 1, \ldots, n-1$ if and only if $\lambda_n|$ is a congruence of geodesics [**1**, pp. 100]. Hence we have

THEOREM 3.4. *The difference between the squares of the variation and the subvariation of a congruence of an orthogonal ennuple with respect to another congruence of the ennuple is the square of the tendency of the former in the direction of the latter. The variation and the subvariation of each of $n-1$ congruences of an orthogonal ennuple with respect to the remaining one congruence of the ennuple are equal if and only if the latter is a congruence of geodesics.*

Let the congruence $\lambda_h|$ of an orthogonal ennuple be normal. Then [**1**, p. 114] we have $\gamma_{h\alpha\beta} = \gamma_{h\beta\alpha}$ ($\alpha, \beta = 1, \ldots, n$; $\alpha \neq \beta \neq h$). Consequently, equation (2.4) reduces to

$$\gamma_{hk}^{\;2} = \gamma_{hkh}^{\;2} + \sum_{m=1}^{n} \gamma_{mhk}^{\;2} = \gamma_{hkh}^{\;2} + 1/r_{hk}^{\;2}, \tag{3.4}$$

since $\gamma_{hhk} = 0$. Hence we have

THEOREM 3.5. *The square of the variation of a normal congruence of an orthogonal ennuple with respect to another congruence of the ennuple differs from the*

square of the angular spread of the same by the square of the tendency of the latter congruence in the direction of the former congruence. The variation and the angular spread of a normal congruence of an orthogonal ennuple with respect to any one congruence of the ennuple are numerically equal if and only if the tendency of the latter in the direction of the former is zero.

Equation (3.4) indicates that the vanishing of γ_{hk} implies the vanishing of γ_{hkh} and $1/r_{hk}$ and conversely. Hence we have

THEOREM 3.6. *If the variation of a normal congruence of an orthogonal ennuple with respect to another congruence of the ennuple is zero, then the former is parallel along the latter and the tendency of the latter in the direction of the former is zero. Conversely, if a normal congruence of an orthogonal ennuple is parallel along another congruence of the ennuple, whose tendency in the direction of the normal congruence is zero, then the variation of the normal congruence with respect to the latter congruence is zero.*

An immediate consequence of the preceding theorem is that *a normal congruence of geodesics of an orthogonal ennuple is parallel along a congruence of the ennuple if and only if the variation of the normal congruence with respect to it is zero.*

By summing over h in (2.4) we obtain

$$\sum_{h=1}^{n} \gamma_{hk}^{2} = \sum_{h=1}^{n} 1/r_{kh}^{2}, \tag{3.5}$$

where $1/r_{kk}$ denotes the first curvature of $\lambda_{k|}$. Note that equation (3.5) holds for general orthogonal ennuple of congruences. Hence we have

THEOREM 3.7. *The curves of a congruence of an orthogonal ennuple are parallel along the curves of all congruences of the ennuple if and only if the variation of the congruence with respect to each congruence of the ennuple is zero. The curves of a congruence of an orthogonal ennuple are geodesics if the variation of the congruence with respect to each congruence of the ennuple is zero.*

If $\gamma_{hk} = 0$ for $h, k = 1, \ldots, n$, then we have from (3.5)

$$1/r_{kh} = 0, \quad 1/r_{kk} = 0 \qquad (h, k = 1, \ldots, n).$$

Consequently, all the congruences of the ennuple consists of geodesics and the curves of each congruence are parallel along the curves of all congruences of the ennuple [**3**, p. 565] and hence parallel along the curves of any congruence in the V_n. Thus we obtain from (3.5)

THEOREM 3.8. *The variation of each congruence of an orthogonal ennuple with respect to every other congruence of the ennuple is zero if and only if the ennuple consists of congruences of geodesics and each congruence of the ennuple is parallel along every other congruence of the ennuple and hence parallel along every congruence in the V_n.*

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A TENSOR EQUATION OF ELLIPTIC TYPE

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The theory of the systems of partial differential equations which arise in connection with the invariant differential operators on a Riemannian manifold may be developed by methods based on those of potential theory. It is therefore natural to consider in the same context the theory of elliptic differential equations, in particular those which are self-adjoint. Some results for a tensor equation in which appears, in addition to the operator Δ of tensor theory, a matrix or double tensor field defined on the manifold, are here presented. The equation may be written

$$\Delta\phi + A\phi = 0,$$

in a notation explained below.

A maximum principle holds for solutions of this equation, under certain conditions on the coefficient tensor A, as is shown in §2. In the following section the construction of de Rham for the Green's form of a closed manifold is extended to this equation, and the solvability of the Poisson equation corresponding is discussed. We then consider Dirichlet and Neumann problems on compact manifolds with boundary, treating first the case when A is positive definite, and then the general case. The necessary integral equation techniques have been developed in [**3**; **4**; **6**a; **6**b; **8**].

1. Definitions. We consider orientable Riemannian manifolds of dimension n and class C^∞. F will denote a closed, compact manifold, M a compact sub-manifold with $(n-1)$-dimensional boundary B also of class C^∞. If a manifold with boundary is given alone, we can define a closed C^∞ manifold (the double) of which the given manifold is a sub-manifold [**4**]. A positive definite metric tensor g_{ij} of class C^∞ is assumed given, and we assume that the curvature of the manifold under consideration is uniformly bounded.

Skew symmetric covariant tensors

$$\phi_{i_1 \dots i_p}$$

of order p on F are associated with extericr differential forms of degree p $(0 < p < n)$;

$$\phi = \phi_{(i_1 \dots i_p)} \, dx^{i_1} \wedge \dots \wedge dx^{i_p}. \tag{1.1}$$

We have the differential operator d, the adjoint $*$, and the co-differential operator $\delta = (-1)^{np+p+1} * d *$. The Laplacian Δ is an elliptic operator defined by

$$\Delta = d\delta + \delta d,$$

that is,

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(1.2) $$(\Delta\phi)_{i_1\ldots i_p} = - D^i D_i \, \phi_{i_1\ldots i_p} + \sum_{\alpha=1}^{p} \Gamma_{j i_1 \ldots i_p}{}^{i(j_1\ldots j_p)} R^h{}_{j_\alpha}{}^j{}_i \phi_{j_1\ldots j_{\alpha-1} h j_{\alpha+1}\ldots j_p}.$$

Here D_i denotes covariant differentiation,

$$\Gamma_{i_1\ldots i_p}{}^{j_1\ldots j_p}$$

is the skew symmetrized Kronecker symbol of order p, and R_{hijk} is the Riemann curvature tensor. Brackets enclosing a group of indices shall mean that summation is to be effected only over those combinations which are in increasing order.

Suppose that we are given on F a tensor field of order $2p$,

$$A = \{A_{i_1\ldots i_p, j_1 \ldots j_p}\},$$

skew-symmetric in each set of p indices, and symmetric in the two sets of indices:

$$A_{i_1\ldots i_p, j_1\ldots j_p} = A_{j_1\ldots j_p, i_1\ldots i_p}.$$

The Ricci tensor R_{ij}, for $p = 1$, and the Riemann tensor $R_{i_1 i_2 j_1 j_2}$, for $p = 2$, are examples of such tensor fields. The components of A are assumed to be of class C^1 in each admissible coordinate system on the manifold. The differential form corresponding to the p-tensor

$$A_{i_1\ldots i_p (j_1 \ldots j_p)} \, \phi^{(j_1\ldots j_p)}$$

will be denoted by $A\phi$. The tensor A will be *non-singular* if there exists a second tensor B such that

$$A_{i_1\ldots i_p (j_1\ldots j_p)} B^{(j_1\ldots j_p) k_1 \ldots k_p} = \Gamma_{i_1\ldots i_p}{}^{k_1\ldots k_p}.$$

Again, A will be termed *positive definite* if the condition

$$A_{(i_1\ldots i_p)(j_1\ldots j_p)} \, \phi^{(i_1\ldots i_p)} \, \phi^{(j_1\ldots j_p)} = 0$$

implies that $\phi^{i_1\ldots i_p}$ is zero. The generalized Kronecker delta is positive definite in this sense. These properties may be interpreted as conditions on the symmetric square matrix of order $\binom{n}{p}$ of independent components of A in any coordinate system. If A is positive definite, A is non-singular. We have

$$*(A\phi) = *A\,(*\phi),$$

where $*A$ is a double skew symmetric tensor density of rank $n - p$:

$$*A_{j_1\ldots j_{n-p}, k_1\ldots k_{n-p}} = e_{(i_1\ldots i_p) j_1 \ldots j_{n-p}} \, e_{(h_1\ldots h_p) k_1 \ldots k_{n-p}} \, A^{(i_1\ldots i_p)(h_1\ldots h_p)}.$$

Here $e_{i_1} \ldots {}_{i_n}$ is the volume n-tensor density.

We introduce the scalar product of two p-tensors

$$(\phi, \psi)_F = \int_F \phi \wedge *\psi = \int_F \psi \wedge *\phi = \int_F \phi_{(i_1\ldots i_p)} \, \psi^{(i_1\ldots i_p)} *1,$$

and the positive definite norm $N(\phi) = (\phi, \phi)_F$. Corresponding to our differential equation

$$\Delta_A \phi = \Delta\phi + A\phi = 0 \tag{1.3}$$

is the Dirichlet integral

$$D(\phi, \psi) = (d\phi, d\psi) + (\delta\phi, \delta\psi) + (\phi, A\psi). \tag{1.4}$$

If A is positive definite, then $D(\phi, \phi)$ is also positive definite. The Green's formula is

$$D(\phi, \psi) - (\phi, \Delta\psi + A\psi) = \int_B [\phi \wedge *d\psi - \delta\psi \wedge *\phi], \tag{1.5}$$

where B is the bounadry of the domain under consideration. On a closed manifold F, if A is positive definite, zero is the only solution of (1.3).

The boundary operators t and n are defined as in [**3**; **4**]; $t\phi$ is the p-form induced on B by the p-form ϕ defined on M, and $n\phi = \phi - t\phi$. The relations $*t = n*$, $*n = t*$ hold.

2. Maximum principle. When $p = 0$, so that (1.3) is a scalar equation and A is a scalar invariant, there holds the following maximum principle: if $A > 0$, no solution has a proper positive maximum, or negative minimum. That is, the square of the solution has no proper maximum. Under analogous conditions, a similar result holds for $p > 0$. The square of ϕ is the invariant

$$\phi^2 = \phi_{(i_1 \ldots i_p)} \phi^{(i_1 \ldots i_p)} > 0. \tag{2.1}$$

We have

$$\begin{aligned} \tfrac{1}{2} D^i D_i \phi^2 &= \tfrac{1}{2} D^i D_i \phi^{(i_1 \ldots i_p)} \phi_{(i_1 \ldots i_p)} \\ &= (D^i \phi^{(i_1 \ldots i_p)})(D_i \phi_{(i_1 \ldots i_p)}) + \phi^{(i_1 \ldots i_p)} D^i D_i \phi_{(i_1 \ldots i_p)}. \end{aligned} \tag{2.2}$$

The first term on the right is non-negative. In view of (1.2) and (1.3), if ϕ is a solution of our differential equation, we may write the second term in the form

$$[A_{(i_1 \ldots i_p)(j_1 \ldots j_p)} + C_{(i_1 \ldots i_p)(j_1 \ldots j_p)}] \phi^{(i_1 \ldots i_p)} \phi^{(j_1 \ldots j_p)},$$

where

$$\begin{aligned} &C_{i_1 \ldots i_p, j_1 \ldots j_p} \\ &\quad = \sum_{n=1}^{p} \Gamma^{i(k_1 \ldots k_p)}{}_{j(i_1 \ldots i_p)} \Gamma_{k_1 \ldots k_{n-1} h k_{n+1} \ldots k_p, j_1 \ldots j_p} R^h{}_{j_n}{}^j{}_i . \end{aligned} \tag{2.3}$$

If therefore the double tensor $A + C$ is positive definite, the quantity $D^i D_i \phi^2$ which appears in (2.2) is positive unless $\phi^2 = 0$. On the other hand, this quantity is non-positive at a maximum of ϕ^2. Consequently ϕ^2 has no maximum value in the interior of any domain in which ϕ is a solution of (1.3).

When a maximum principle holds in this form, (1.3) has no non-zero solutions regular on a closed manifold F, and furthermore the solution of the Dirichlet problem on a manifold with boundary is unique. If C is positive definite, this leads to an improvement of the results which can be obtained by use of the Dirichlet integral. Sets of conditions under which C is positive definite have been

formulated by Lichnerowicz [7]. We remark that in a space of constant curvature K, $C = nK\Gamma$ is positive definite if K is positive. When $p = 1$, $C_{ij} = R_{ij}$.

To compare the limitations obtained in a closed manifold by the Dirichlet integral and the maximum principle, consider the following example with $p = 1$:

$$(\Delta\phi)_i + \lambda R_i^{\ j}\, \phi_j = 0, \qquad \lambda \text{ real};$$

or,

$$D^k D_k \phi_i + (1 - \lambda)\, R_i^{\ j} \phi_j = 0.$$

The Dirichlet integral shows that if $-R_{ij}$ is positive definite (positive mean curvature) there are no solutions for $\lambda < 0$ and if R_{ij} is positive definite, no solutions for $\lambda > 0$. The maximum principle shows in the first case ($-R_{ij}$ positive definite) there are no solutions if $\lambda < +1$, which is an improvement; but in the other case shows only that there are no solutions if $\lambda > +1$. We remark that a Killing vector ξ_i, with $D_j\xi_i + D_i\xi_j = 0$, satisfies the above equation with $\lambda = +2$. If R_{ij} is positive definite, there are no Killing vectors, as is shown [2] by both methods.

3. The Green's form. The method used by de Rham to construct the Green's form for Laplace's equation on a closed manifold carries over with but minor alterations to the equation (1.3). For completeness, however, we will describe the construction [8].

We assume that there exists a positive number η such that if x and y are any two points at a distance less than η, a unique geodesic can be drawn from x to y. Let $s(x, y)$ be the geodesic distance so defined, and set

$$a_{i,j} = -\frac{1}{2}\frac{\partial^2 s(x, y)}{\partial x^i \partial y^j}.$$

Let

$$a_{i_1 \dots i_p, j_1 \dots j_p}$$

be the determinant

$$| a_{i_\rho j_\sigma} |, \qquad 1 \leq \rho, \sigma \leq p.$$

Let $\rho(x, y)$ be a function of class C^∞, $\rho = \rho(s(x, y)) = 1$ for $s(x, y) < \frac{1}{2}\eta$, $\rho = 0$ for $s(x, y) \geq \eta$. If ω_n denotes the area of the unit sphere in E_n, we define the parametrix

$$(3.1)\quad \omega(x, y) = \omega_p(x, y) = \frac{s^{-n+2}(x, y)\, \rho(x, y)}{(n-2)\,\omega_n} a_{(i_1 \dots i_p)(j_1 \dots j_p)}\, dx^{i_1} \wedge \dots \wedge dx^{i_p}\, dy^{j_1} \wedge \dots \wedge dy^{j_p}.$$

For $n = 2$, $s^{-n+2}/(n - 2)$ is to be replaced by $(-\log s)$.

The integral operator Ω with kernel $\omega = \omega_p$ has the following properties [8]:

(1) $\Omega\phi(x) = (\omega(x, y), \phi(y))_p$ is of class C^∞ if ϕ is of class C^∞.

(2) Ω is self-adjoint since ω is symmetric.

(3) $q(x, y) = \Delta_x(x, y)$ is $O(s^{-n+2})$ as $s \to 0$.

We set

$$Q_A\phi(x) = (q(x, y) + A(x)\,\omega(x, y), \phi(y))_F \tag{3.2}$$

and

$$Q'_A\phi(x) = (q(y, x) + A(y)\,\omega(y, x), \phi(y))_F. \tag{3.3}$$

It then follows that

$$\Omega\Delta_A\phi = \Omega\Delta\phi + \Omega A\phi = \phi + Q'_A\phi \tag{3.4}$$

and

$$\Delta_A\Omega\phi = \Delta\Omega\phi + A\Omega\phi = \phi + Q_A\phi. \tag{3.5}$$

Consider the non-homogeneous equation

$$\Delta\psi + A\psi = \rho, \tag{3.6}$$

where ρ is a given p-form of class C'. Let E denote the linear space of solution of (1.3) regular in F. If $\phi \in E$, we have from (3.4),

$$\phi + Q'_A\,\phi = 0, \tag{3.7}$$

and this equation has only a finite number of linearly independent solutions, so that E is finite dimensional. For $\phi \in E$ we also have

$$(\rho, \phi) = (\Delta\psi + A\psi, \phi) = (\psi, \Delta\phi + A\phi) = 0.$$

A necessary condition for the solvability of (3.6) is therefore that ρ be orthogonal to E.

To solve (3.6) set $\psi = \Omega\xi$, so that (3.5) yields the integral equation

$$\xi + Q_A\xi = \rho. \tag{3.8}$$

A solution ξ exists if and only if ρ is orthogonal to all solutions of the homogeneous transposed equation (3.7). We must necessarily assume that ρ is orthogonal to E; let E_1 denote the orthogonal complement of E in the space of solutions of (3.7). The following device of de Rham shows that (3.6) may still be solved even if ρ is not orthogonal to E_1 but only to E. If $\phi_1 \in E_1$, $\phi_1 \not\equiv 0$, then

$$((\Delta + A)^2\phi, \rho_1) \not\equiv 0$$

for every $\phi \in E_1$, since

$$((\Delta + A)^2\phi_1, \phi_1) = N((\Delta + A)\,\rho_1) > 0$$

since ϕ_1 is orthogonal to E and is not zero. To each ϕ_1 in E_1 corresponds a linear functional $((\Delta + A)^2\phi_1, \phi)$ defined for $\phi \in E_1$, which functional does not vanish identically if $\phi_1 \not\equiv 0$. Conversely therefore, each linear functional on the finite-dimensional space E_1 can be represented in the form $((\Delta + A)^2\phi_1, \phi)$ for a suitable ϕ_1. Thus there exists a ϕ_1 with

$$(\rho, \phi) = ((\Delta + A)^2\phi_1, \phi), \qquad \rho \in E.$$

It follows that $\rho - (\Delta + A)^2\phi_1$ is orthogonal to E_1, and also to E since each term separately is orthogonal to E. The integral equation

$$\xi - Q_A\xi = \rho - (\Delta + A)^2\phi_1$$

therefore has a solution ξ, and now ψ, defined by

$$\psi = \Omega\xi + (\Delta + A)\phi_1$$

is a solution of (3.6). We therefore have

THEOREM 1. *On a closed Riemannian space there are at most a finite number of linearly independent solutions of $\Delta\phi + A\phi = 0$. If A, or $A + C$, is positive definite, zero is the only solution. The non-homogeneous equation $\Delta\phi + A\phi = \rho$ has a solution if and only if ρ is orthogonal to all solutions of the homogeneous equation.*

Let $\alpha_p(x, y)$ be the reproducing kernel of the space E, and let

$$E\phi = (\alpha, \phi)_F \tag{3.9}$$

be the projection of an arbitrary form ϕ upon E. By Theorem I the equation

$$\Delta\phi + A\phi = \rho - E\rho \tag{3.10}$$

has a solution ϕ. Furthermore it has a unique solution orthogonal to E. Let this solution be denoted by $G\rho = G_A\rho$. It follows as in (8) that G_A is an integral operator whose kernel, which we denote by $g_A(x, y)$ is symmetric, has the singularity of a local fundamental singularity for (1.3), and satisfies

$$(\Delta + A)_x g_A(x, y) = -\alpha(x, y), \qquad x \neq y. \tag{3.11}$$

If A is positive definite, $\alpha \equiv 0$ since E is zero, and $g_A(x, y)$ is a fundamental solution in the large for equation (1.3).

4. Boundary value problems. We shall now assume that the double tensor A is defined and positive definite on a compact manifold M with boundary B. Let us enlarge M by adjoining to M a neighbourhood $B \times I$ of the boundary B with a fixed closed interval I, B being identified with an endpoint of I. If we call this enlarged finite manifold M_1, the double F_1 of M_1 consists of M_1 and an oppositely oriented replica of M_1, with corresponding boundary points identified. Clearly A is defined in a natural way on F_1 except in the replica of the boundary neighbourhood. By a suitable averaging or interpolation, A can now be defined in the combined boundary neighbourhoods so as to be positive definite and of the same degree of regularity (up to and including C^∞) as in M. We may therefore regard A as defined in F, a closed manifold of which M is a sub-manifold. Similarly the metric tensor g_{ij} can be extended to F, remaining positive definite and of class C^∞. Since A is positive definite on F, the Green's form $g_A(x, y)$ of F is a fundamental singularity in the large for (1.3) on M.

The Dirichlet problem consists of finding a solution ϕ of (1.3) with $t\phi$, $t{*}\phi$ taking assigned continuous boundary values on B. For the Neumann problem the assigned data are $t{*}d\phi$, $t{*}d{*}\phi$. Since the Dirichlet integral for M is positive definite, it follows at once that solutions of these problems are unique.

We construct double and single layer potentials [**3**]:

$$\mu(x) = \int_B [\rho(y) \wedge *dg_A(x, y) + \rho(y) \wedge *d*g_A(x, y)], \tag{4.1}$$

$$\nu(x) = \int_B [g_A(x, y) \wedge *d\sigma(y) + *g_A(x, y) \wedge *d*\sigma(y)] \tag{4.2}$$

with densities ρ and σ respectively, on B. The singularity of $g_A(x, y)$ is, in its principal term, the same as the singularity of the de Rham form $g(x, y)$. Consequently the regularity behaviour and discontinuities of (4.1) and (4.2) are the same as for the corresponding potentials in [**3**]. As the argument point passes from M across B, the quantities

$$t\mu, \quad t*\mu, \quad t*d\nu, \quad t*d*\nu$$

have the respective discontinuities

$$t\rho, \quad t*\rho, \quad -t*d\sigma, \quad -t*d*\sigma.$$

On the boundary B we have, as limiting values from the interior of M,

$$\begin{aligned} t_-\mu &= \tfrac{1}{2}t\rho + t\int_B (\rho \wedge *dg_A + *\rho \wedge *d*g_A), \\ t_-*\mu &= \tfrac{1}{2}t*\rho + t*\int_B (\rho \wedge *dg_A + *\rho \wedge *d*g_A), \\ t_-*d\nu &= -\tfrac{1}{2}t*d\sigma + t*d\int_B (g_A \wedge *d\sigma + *g_A \wedge *d*\sigma), \\ t_-*d*\nu &= -\tfrac{1}{2}t*d*\sigma + t*d*\int_B (g_A \wedge *d\sigma + *g_A \wedge *d*\sigma). \end{aligned} \tag{4.3}$$

The integrals on the right are to be interpreted as principal values. To obtain limiting values on B from the complement CM of M in F, the signs of the leading terms on the right should be reversed. Limiting values from CM will be indicated by a subscript $+$ sign.

The solution of the Dirichlet problem is therefore to be obtained by solving the integral equations

$$t_-\mu = t\phi, \quad t_-*\mu = t*\phi, \tag{4.4}$$

where $t\phi$, $t*\phi$ are the assigned continuous boundary values. In analogous fashion, the Neumann problem may be solved by means of the system.

$$t_-*d\nu = t*d\phi, \quad t_-*d*\nu = t*d*\phi, \tag{4.5}$$

for given continuous $t*d\phi$, $t*d*\phi$ on B. These are systems of $\binom{n}{p}$ singular integral equations. Since $g_A(x, y) = g_A(y, x)$, it can be shown as in [**3**] that the kernels of the systems (4.4) and (4.5) are transposes of each other.

The condition for the existence of a solution is that the assigned non-homogeneous term be orthogonal to every solution of the homogeneous transposed equation [**5**]. In each case the homogeneous transposed equation is obtained when we attempt to solve the boundary value problem of the complementary type for the complementary domain CM.

For the Dirichlet problem we must therefore show that

$$\int_B [\phi \wedge *d\sigma + *\phi \wedge *d*\sigma] = 0 \tag{4.6}$$

for all solutions σ of the equations

$$t_+*d\nu = 0, \quad t_+*d*\nu = 0.$$

Now (4.7) imply $\nu \equiv 0$ in CM. Since ν is continuous across B, we have $t_-\nu = 0$, $t_-*\nu = 0$, and therefore $\nu \equiv 0$ in M. Therefore the discontinuity conditions, which read

$$t_-*d\nu = t*d\sigma, \qquad t_-*d*\nu = t*d*\sigma,$$

imply that $t*d\sigma$ and $t*d*\sigma$ vanish. Thus (4.6) holds for all continuous $t\phi$, $t*\phi$.

THEOREM II. *If A is positive definite in M, there exists a unique solution of the Dirichlet problem for $\Delta\phi + A\phi = 0$, with $t\phi$, $t*\phi$ having given continuous values on B.*

The condition of solvability for the Neumann problem is

$$\int_B [\rho \wedge *d\phi + *\rho \wedge *d*\phi] = 0, \tag{4.8}$$

for all solutions ρ of the equations

$$t_+\mu = 0, \quad t_+*\mu = 0. \tag{4.9}$$

These conditions imply that μ vanishes identically in CM It follows as in [**3**] that $t*d\mu$ and $t*d*\mu$ are continuous across B, and therefore

$$t_-*d\mu = 0, \quad t_-*d*\mu = 0.$$

It follows now that $\mu = 0$ in M. The discontinuity conditions show that

$$t_-\mu = -t\rho, \quad t_-*\mu = -t*\rho,$$

and therefore $t\rho$ and $t*\rho$ are zero. Thus (4.8) is always satisfied.

THEOREM III. *If A is positive definite in M, there exists a unique solution of the Neumann problem for $\Delta\phi + A\phi = 0$, with $t*d\phi$, $t*d*\phi$ having given continuous values on B.*

The Green's and Neumann's forms for M corresponding to the Dirichlet and Neumann problems are easily defined by subtracting a regular solution, suitably determined, from the fundamental solution $g_A(x, y)$. Representation formulae for the solutions of the Dirichlet and Neumann problems follow in the usual way from Green's formula (1.5). We note that the boundary conditions are

$$t\,G_A(x, y) = 0, \quad t*G_A(x, y) = 0, \quad t*dN_A(x, y) = 0, \quad t*d*N_A(x, y) = 0, \tag{4.10}$$

and that $G_A(x, y)$ and $N_A(x, y)$ are symmetric in their two arguments. As in the scalar theory, the difference

$$K_A(x, y) = N_A(x, y) - G_A(x, y) \tag{4.11}$$

is the reproducing kernel for solutions of (1.3) in the D-metric (1.4) over M. The kernel (4.11) has many of the formal properties developed in [**1**].

5. Boundary value problems (continued). We now consider the Dirichlet and Neumann problems for M when the coefficient tensor A is not restricted to be positive definite. It is then possible that the solutions of the boundary value problems will, if they exist, not be unique. We require A to be of class C^1 in $M + B$. Consider the equation (1.3) in the form

$$\Delta\phi + A_0\phi = (A_0 - A)\phi = B\phi, \tag{5.1}$$

say, where A_0 is yet to be chosen. We shall need to choose A_0 positive definite and such that $B = A_0 - A$ is non-singular. Let A_1 be any positive definite double tensor of the type of A.

The characteristic roots of the matrix of independent components of A_1 in any coordinate system are real, since A_1 is assumed symmetric, and positive, since A_1 is positive definite. Also the characteristic roots of $A_1 - \epsilon A$ are continuous functions of ϵ, and of the point where $A_1 - \epsilon A$ is evaluated. For ϵ sufficiently small, and positive, the roots are positive at any point of M. Since M is compact, it follows that there is an $\epsilon_1 > 0$ such that the roots of $A_1 - \epsilon_1 A$ are positive everywhere in $M + B$. We now choose $A_0 = \epsilon_1^{-1}A_1$; $B = \epsilon_1^{-1}A_1 - A$ is positive definite and therefore non-singular.

Denote by $G_0(x, y)$ and $N_0(x, y)$ the Green's and Neumann's tensors of M for the equation

$$\Delta\phi + A_0\phi = 0.$$

We see that any solution ϕ of the Dirichlet problem for (1.3) must satisfy the integral equation

$$\phi(x) + (B\phi(y), G_0(x, y))_M = \psi(x), \tag{5.2}$$

where $\Delta\psi + A_0\psi = 0$, and $t\psi$, $t*\psi$ assume the given boundary values of the problem. If we apply the operator $\Delta + A_0$ to (5.2), we find, conversely,

$$\Delta\phi + A_0\phi - B\phi = 0,$$

so that ϕ is a solution of (5.1), that is to say, of (1.3). Also, on taking boundary values we note that the integrated term yields zero because of the boundary behaviour of the Green's function, and therefore that $t\phi = t\psi$, $t*\phi = t*\psi$. If $t\psi$, $t*\psi$ are assigned continuously, ψ is uniquely determined, by Theorem II, and any solution of (5.2) is a solution of the Dirichlet problem for (1.3).

A solution of (5.2) exists if and only if ψ is orthogonal to all solutions χ of the homogeneous transposed equation

$$\chi(x) + B(x)\,(\chi(y), G_0(x, y))_M = 0. \tag{5.3}$$

Since B is non-singular, B^{-1} exists, and we have

$$\begin{aligned} 0 &= B^{-1}\chi(x) + (BB^{-1}\chi(y), G_0(x, y))_M \\ &= B^{-1}\chi(x) + (B^{-1}\chi(y), BG_0(x, y))_M. \end{aligned}$$

Therefore the tensor $B^{-1}\chi$ is a solution of the homogeneous equation corresponding to (5.2):

$$\phi(x) + (B\phi(y),\ G_0(x, y))_M = 0. \tag{5.4}$$

This equation has continuous iterated kernels of sufficiently high order, and therefore a finite number of linearly independent solutions $\phi_r(r = 1, \dots, N)$. Thus any solution χ of (5.3) is of the form $\chi = B\phi$ for some solution $\phi = \sum a_r\phi_r$ of (5.4). The condition of solvability of (5.2) is therefore $(\chi, \psi) = (B\phi, \psi) = 0$ for every solution p of (5.4). We have from (1.5) and (5.4)

$$\begin{aligned}(\chi, \psi) = (B\phi, \psi) &= (\Delta\phi + A_0\,\phi, \psi) \\ &= (\phi, \Delta\psi + A_0\psi) + \int_B [\phi \wedge *d\psi - \delta\psi \wedge *\phi \\ &\qquad - \psi \wedge *d\phi + \delta\phi \wedge *\psi].\end{aligned}$$

The volume integral vanishes since $\Delta\psi + A\psi = 0$, and $t\phi = 0$, $t{*}\phi = 0$ since ϕ is a solution of (5.4). We therefore have

THEOREM IV. *There exists a solution of the Dirichlet problem for $\Delta\phi + A\phi = 0$, $t\phi = t\psi$, $t{*}\phi = t{*}\psi$ if and only if*

$$\int_B [\psi \wedge *d\phi_r - \delta\phi_r \wedge *\psi] = 0 \tag{5.5}$$

for every solution ϕ_r of $\Delta\phi + A\phi = 0$, $t\phi = 0$, $t{*}\phi = 0$.

The condition (5.5) involves only the given data $t\psi$, $t{*}\psi$, and the eigentensors ϕ_r. The most general solution of the problem is of course of the form

$$\phi + \sum a_r\phi_r$$

where ϕ is any particular solution.

The Neumann problem may also be treated this way. Any solution of the problem must satisfy the integral equation

$$\phi(x) + (B\phi(y),\ N_0(x, y))_M = \psi(x), \tag{5.6}$$

where $\psi(x)$ is that solution of the equation $\Delta\psi + A_0\psi = 0$ with the assigned values of $t{*}d\psi$, $t{*}d{*}\psi$. Conversely, a solution of (5.6) provides a solution of the Neumann problem. A solution of (5.6) exists if and only if ψ is orthogonal to all solutions of the homogeneous transposed equation

$$\chi(x) + B(x)(\chi(y), N_0(x, y))_M = 0. \tag{5.7}$$

As before, $\chi = B\phi_r$ where ϕ_r is a solution of (5.6) with $\psi = 0$. The orthogonality condition can be transformed to read

$$\begin{aligned}0 = (\psi, \chi) = (\psi, B\phi) &= (\Delta\phi + A_0\phi, \psi) \\ &= (\phi, \Delta\psi + A_0\psi) + \int_B (\phi \wedge *d\psi - \delta\psi \wedge *\phi \\ &\qquad - \psi \wedge *d\phi - \delta\phi \wedge *\psi).\end{aligned}$$

The volume integral and two terms of the surface integral vanish, as before.

THEOREM V. *There exists a solution of the Neumann problem for* $\Delta\phi + A\phi = 0$, $t{*}d\phi = t{*}d\psi$, $t{*}d{*}\phi = t{*}d{*}\psi$ *if and only if*

$$\int_B (\phi_r \wedge {*}d\psi - \delta\psi \wedge {*}\phi_r) = 0 \tag{5.8}$$

for every solution ϕ_r *of* $\Delta\phi + A\phi = 0$, $t{*}d\phi = 0$, $t{*}d{*}\phi = 0$.

The form of the general solution is obvious.

We remark that the third, or mixed, boundary value problem of potential theory for the equation (1.3) may also be treated this way. The indirect method used here circumvents some of the characteristic difficulties of the corresponding proofs for the Laplace equation, such as the lack of a fundamental singularity in the large.

6. The Laplace equation. The methods used in the foregoing work enable us to give a quite short proof of the following result on the existence of a fundamental singularity in the large for the tensor Laplace equation on M:

THEOREM VI. *There exists a fundamental singularity in the large on* M *for* $\Delta\phi = 0$ *if and only if the Dirichlet problem for* $\Delta\phi = 0$ *on* M *has at most one solution.*

We first show that, if the Dirichlet problem has at most one solution, the fundamental singularity exists. Let F be the double of M, and let us extend the differential equation to F in the form

$$\Delta\phi + A\phi = 0, \tag{6.1}$$

where A is, as before, a matrix or double p-tensor such that A is C^∞ in F, positive definite in $F - M$, and zero in M itself. For instance, the Kronecker tensor Γ multiplied by a suitable scalar factor provides such a tensor. Then we construct the Green's form of F for (6.1); from (3.11) we see that this Green's form will itself be a fundamental singularity in the large for (6.1), provided only that there exist no everywhere regular solutions of (6.1) in F, except zero. If such a regular solution did exist, it would have a zero Dirichlet integral over F. Since A is positive definite in $F - M$, the solution must be zero there, and by continuity it must be zero on B. Hence it is harmonic in M, and has zero boundary value; by hypothesis it must be identically zero. Thus the Green's form does provide the desired fundamental singularity.

The converse part of the theorem may be established as follows. Assume that a fundamental singularity exists in M for the Laplace operator, then we may use it to solve the Poisson equation $\Delta\phi = \beta$ for arbitrary $\beta \in C^3$ in M. From Green's formulae (1.4) and (1.5) with $A = 0$, we see that the conditions $\Delta\phi = 0$ in M, with $t\phi = 0$, $t{*}\phi = 0$ on B, imply that $d\phi = 0$, $\delta\phi = 0$ in M. From Theorem IV we see that there exists a solution in M of the Laplace equation having arbitrary continuous boundary values. Suppose now that ϕ is

a solution of $\Delta\phi = 0$ with $t\phi = 0$, $t{*}\phi = 0$. In view of the remarks just made, there exists a p-tensor ψ with $\Delta\psi = \phi$ in M, $t\psi = 0$, $t{*}\psi = 0$ on B. Then

$$N(\phi) = (\phi, \Delta\psi)$$

$$= (\psi, \Delta\phi) + \int_B [\phi \wedge *d\psi - \delta\psi \wedge *\phi - \psi \wedge *d\psi + \delta\phi \wedge *\psi] = 0$$

since all terms on the right vanish. Hence $\phi \equiv 0$, which shows that a solution of the Dirichlet problem for M is unique. This completes the proof.

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NULL TRIGONOMETRIC SERIES IN DIFFERENTIAL EQUATIONS

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1. Introduction. In this paper it is shown how trigonometric series which are Cesàro-summable to zero may be used to solve differential equations. The explicit solution of the general ordinary linear equation with constant coefficients is found in terms of trigonometric series and special cases are dealt with.

2. Null trigonometric series. By the Heine-Cantor and subsequent theorems, if the trigonometric series

2.1 $$\tfrac{1}{2} a_0 + \sum (a_n \cos nx + b_n \sin nx) = \sum c_n e^{inx},$$

where $\sum$ applied to the real form denotes summation for n from 1 to ∞ and applied to the complex form from $-\infty$ to ∞, is convergent to the sum zero for all values of x in the closed interval $(-\pi, \pi)$, or for all values except (possibly) those of a uniqueness (or unicity) set, then every one of the coefficients a_n, b_n, c_n must be zero [**6**, p. 103; **9**, pp. 274, 291].[1]

If convergence to zero is replaced by summability to zero, say by the Abel-Poisson or the Cesàro definition, the position is different. There are series with non-vanishing coefficients which have the sum zero for all values of x, with or without exceptional values [**9**, p. 297].

A trigonometric series whose coefficients are not all zero, whose sum by a method (T) is zero for all values of x, or for all values with specified exceptions, may be called a *null trigonometric series*, or briefly NTS, in the field T. In what follows T will be Cesàro summation to some positive integral order, specified or unspecified.

The simplest examples of NTS are

2.2 $$\tfrac{1}{2} + \sum \cos nx = \tfrac{1}{2} \sum e^{inx},$$

2.3 $$\sum n \sin nx = -\tfrac{1}{2} \sum ine^{inx},$$

which are respectively summable (C, 1) to zero for $x \not\equiv 0 \pmod{2\pi}$ and summable (C, 2) to zero for all values including $x \equiv 0$, the summability of 2.3 being non-uniform in the neighbourhood of $x \equiv 0$.

More generally, using D to denote d/dx, the series

2.4 $$D^r \tfrac{1}{2} + \sum D^r \cos n(x - \alpha) = \tfrac{1}{2} \sum (in)^r e^{in(x-\alpha)},$$

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[1]There appears to be a slight error of statement in [**6**, p. 104, Osservazione II].

where r is any non-negative integer and α any real number, is null (C, $r+1$) for all real values of x except $x \equiv \alpha \pmod{2\pi}$ if r is even or zero, and for all values without exception if r is odd. The series 2.4 has a "singularity" at $x \equiv \alpha$, where it ceases to be finitely summable if r is even and the summability is non-uniform if r is odd [**2**, p. 2].

It may be observed that these series are the expansions of the "Dirac function" and its derivatives [**5**].

From the work of Verblunsky and others it follows that if the series 2.1 is summable (A) to zero for $x \not\equiv 0$ and the condition $|a_n| + |b_n| = o(n)$ is satisfied then the series is a constant multiple of series 2.2. Since this condition is necessary for (C, 1) summability, the same applies to (C, 1), and it follows that the only trigonometric series which are null (C, 1) with the single singularity at $x \equiv \alpha$ are constant multiples of the series 2.4 with $r = 0$. If there are a finite number of singularities, say $\alpha_1, \alpha_2, \ldots, \alpha_m$, in each period, the series are linear combinations of series of the same type with $\alpha = a_1, \alpha_2, \ldots, \alpha_m$.

For (C, k) summability the work of Wolf shows that the null series are linear combinations of series of type 2.4 where the index r has the values $0, 1, \ldots, k-1$ [**7**; **9**, p. 302; **4**, p. 92; **8**, p. 355; **1**].

If there is only one singularity α in a period and $\alpha \equiv \pi$, the null (C, k) series must be a linear combination of series of type 2.4 with $r = 0, 1, 2, \ldots, k-1$, and $\alpha \equiv \pi$, viz.

2.5
$$\Lambda_k(-\pi, \pi) = \sum_{r=1}^{k} A'_r[D^{r-1}\tfrac{1}{2} + \sum (-1)^n D^{r-1} \cos nx]$$
$$= \sum_{r=1}^{k} A_r \sum (-1)^n n^{r-1} e^{inx}$$

where the A'_r, A_r are arbitrary constants. This may be called the general NTS of order k for $(-\pi, \pi)$.

3. Solution of differential equations in trigonometric series. Let

3.1
$$\Phi = 0$$

be a differential equation. In what follows only ordinary equations of a sufficiently simple type will be considered, so that Φ represents a function of x, y, and the derivatives of y. In the case of a linear equation of order m with constant coefficients we shall have $\Phi = F(D)y - f(x)$, where $D = d/dx$, $F(D)$ is a polynomial of degree m with constant coefficients and $f(x)$ is a function of x.

Our purpose is to solve 3.1 in periodic form, getting the general solution (complete primitive) in terms of trigonometric series and involving arbitrary constants. For special solutions incorporating given "initial conditions" it may be observed that for a problem of periodic type such conditions may involve the average value of y, or the general property of continuity and periodicity, or other conditions which lead easily to the evaluation of the constants. An example of such a special solution is 3.8(b) below.

Consider any trigonometric series

3.2 $$\tfrac{1}{2} a_0 + \sum (a_n \cos nx + b_n \sin nx) = \sum c_n e^{inx}$$

and its rth derived series

3.3 $$\sum n^r [a_n \cos (nx + \tfrac{1}{2} r\pi) + b_n \sin (nx + \tfrac{1}{2} r\pi)] = \sum (in)^r c_n e^{inx}.$$

Suppose 3.3, for $r = m$, is summable (C, k_1), where k_1 is a non-negative integer; and suppose that the summability is uniform over any closed interval $-\pi < -\omega < x < \omega < \pi$, interior to $(-\pi, \pi)$. It follows, by a theorem on convergence factors [**2**, p. 131, Theorem 76], that 3.3 is necessarily summable (C, $k_1 - m + r$) for $r = 0, 1, \ldots, m - 1$ provided $k_1 - m + r > 0$. Thus all the series 3.2, 3.3 for $r = 1, 2, \ldots, m$ are uniformly summable (C, k_1). Then, by a theorem on differentiation of summable series [**2**, p. 349, Theorem 249], if the sum of 3.2 is a function y having derivatives $D^r y$ for $r = 1, 2, \ldots, m$ then the (C, k_1) sums of the series 3.3 are equal to these derivatives; thus

3.4 $$D^r y = \sum n^r [a_n \cos (nx + \tfrac{1}{2} r\pi) + b_n \sin (nx + \tfrac{1}{2} r\pi)] = \sum (in)^r c_n e^{inx}, \quad (\mathrm{C}, k_1)$$

for $r = 1, 2, \ldots, m$.

If now $F(D)$ is a differential operator of order m, of suitable type, we shall have

$$F(D)y = F(D)\tfrac{1}{2}a_0 + \sum F(D)(a_n \cos nx + b_n \sin nx)$$

3.5 $$= \tfrac{1}{2}a'_0 + \sum (a'_n \cos nx + b'_n \sin nx)$$

$$= \sum F(D)c_n e^{inx} = \sum c'_n e^{inx}, \qquad (\mathrm{C}, k_2),$$

for some integral value of k_2. This will necessarily be the case if $F(D)$ is a polynomial with constant coefficients; then $c'_n = F(in)c_n$, with corresponding values for a'_n, b'_n; and $k_2 = k_1$.

If then $f(x)$ is a function which can be expanded in $(-\pi, \pi)$ as a trigonometric series, summable (C, k_3), say

3.6 $$f(x) = \tfrac{1}{2}\alpha_0 + \sum (\alpha_n \cos nx + \beta_n \sin nx) = \sum \gamma_n e^{inx}, \qquad (\mathrm{C}, k_3),$$

the differential equation 3.1 with $\Phi = F(D)y - f(x)$ will be equivalent to

3.7 $$\tfrac{1}{2}(a'_0 - \alpha_0) + \sum [(a'_n - \alpha_n) \cos nx + (b'_n - \beta_n) \sin nx] = 0$$

or

$$\sum (c'_n - \gamma_n) e^{inx} = 0, \qquad (\mathrm{C}, k),$$

where k is some integer, viz. the greater of k_2, k_3.

By the Heine-Cantor theory, described in §2, the equality 3.7 could be satisfied for all x in $(-\pi, \pi)$ with the possible exception of $x \equiv \pi$ in the field of convergence (i.e. with $k = 0$) by and only by equating to zero the coefficients $a'_n - \alpha_n$, $b'_n - \beta_n$, or $c'_n - \gamma_n$, for all integral n. The coefficients a_n, b_n, or c_n might thence be deduced and so give a solution of 3.1; but the general solution could not usually be found in this way.

By the Verblunsky-Wolf theory, however, the equality 3.7 will be satisfied in the field (C, k) where $k > 1$ by equating the coefficients to the corresponding coefficients of the general NTS of order k with singularity at $x \equiv \pi$, viz. $\Lambda_k(-\pi, \pi)$ of 2.5. The corresponding coefficient-equations are

$$a'_0 - \alpha_0 = A'_1, \quad a'_n - \alpha_n = (-1)^n(A'_1 - n^2 A'_3 + \ldots \pm n^{2s} A'_{2s+1}),$$

$$b'_n - \beta_n = (-1)^n(-nA'_2 + n^3 A'_4 - \ldots \pm n^{2t-1} A'_{2t});$$

$$c'_n - \gamma_n = (-1)^n(A_1 + n A_2 + \ldots + n^{k-1} A_k);$$

where s, t are the greatest integers for which $2s - 1 < k$, $2t < k$. These equations will, in suitable cases, determine the constants a'_n, b'_n, or c'_n, and thence a_n, b_n, or c_n. The required solution will then be found. If the solution, when found, is such that its derived series of order m is Cesàro-summable, uniformly in $-\pi < -\omega \leq x \leq \omega < \pi$, the process is justified and the solution will be the true solution.

To elucidate the process, consider the equations:

3.8 (a) $(D^2 + q^2)\, y = 0$, (b) $(D^2 + q^2)\, y = \frac{1}{2} \cot \frac{1}{2}x$,

where q is real but not integral, finding for (b), besides the general solution, the special solution with initial conditions:

(i) y is continuous (including $x \equiv \pi$) and periodic 2π.

(ii) The mean value of y over a period is y_0.

Using the real form, let $y = \frac{1}{2}a_0 + \sum(a_n \cos nx + b_n \sin nx)$. The equation (a) then gives

$$q^2\, \tfrac{1}{2}\, a_0 + \sum\, (q^2 - n^2)(a_n \cos nx + b_n \sin nx) = 0.$$

If we equated the coefficients to zero, treating this series as a convergent trigonometric series, we should obtain only the useless particular solution $y = 0$. So equate the coefficients instead to those of a suitable NTS. Taking the NTS $\Lambda_k(-\pi, \pi)$ of 2.5 with $k = 2$, viz.

$$A_1[\tfrac{1}{2} + \sum\, (-1)^n \cos nx] + A_2 \sum\, (-1)^n\, n \sin nx,$$

the coefficient equations are

$$q^2\, \tfrac{1}{2}a_0 = \tfrac{1}{2}A_1, \quad (q^2 - n^2)\, a_n = A_1(-1)^n, \quad (q^2 - n^2)\, b_n = A_2 n(-1)^n.$$

The solution of (a) is therefore

$$y = A_1\left[\frac{1}{2q^2} + \sum \frac{(-1)^n \cos nx}{q^2 - n^2}\right] + A_2 \sum \frac{(-1)^n\, n \sin nx}{q^2 - n^2}.$$

This is convergent for all x, but non-uniformly so[2] in the neighbourhood of $x \equiv \pi$ unless $A_2 = 0$. To prove the validity of the process in this case it suffices

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[2]The choice of Λ with k equal to the order of the equation is governed by the fact that any greater value of k would lead only to a solution Cesàro-equivalent to the one found.

to observe that the second derived series of the series found for y is uniformly summable (C, 2) in $(-\pi < -\omega < x < \omega < \pi)$ in virtue of the equalities

$$\sum \frac{(-1)^n(-n^2)\cos nx}{q^2-n^2} = \sum (-1)^n \cos nx - q^2 \sum \frac{(-1)^n \cos nx}{q^2-n^2},$$

$$\sum \frac{(-1)^n(-n^3)\sin nx}{q^2-n^2} = \sum (-1)^n n \sin nx - q^2 \sum \frac{(-1)^n n \sin nx}{q^2-n^2}.$$

It is easy to verify the solution otherwise by finding the corresponding Fourier series for the usual form of solution in terms of $\cos qx$, $\sin qx$.

To solve (b) we have $\frac{1}{2}\cot\frac{1}{2}x = \sum \sin nx$, (C, 1) (this being the Cauchy principal-value Fourier series of $\frac{1}{2}\cot\frac{1}{2}x$) and we have only to subtract the coefficient 1 of $\sin nx$ in this series from b_n in the above coefficient-equations. The resulting general solution of (b) is

$$y = A_1\left[\frac{1}{2q^2} + \sum \frac{(-1)^n \cos nx}{q^2-n^2}\right] + A_2 \sum \frac{(-1)^n n \sin nx}{q^2-n^2} + \sum \frac{\sin nx}{q^2-n^2}.$$

This is convergent for all x, but non-uniformly about $x \equiv \pi$. In this case, in consequence of the point of non-uniform summability of $\sum \sin nx$ and infinite discontinuity of $\frac{1}{2}\cot\frac{1}{2}x$ at $x \equiv 0$, the second derived series is non-uniformly summable about $x \equiv 0$ as well as $x \equiv \pi$; but the justification applies to the two open intervals $(-\pi, 0)$, $(0, \pi)$ separately.

To find the required special solution, condition (i) shows that $A_2 = 0$ because the sum of the series $\sum(-1)^n n(q^2 - n^2)^{-1}\sin nx$ is discontinuous at $x \equiv \pi$ like $\sum (-1)^n n^{-1} \sin nx$, and (ii) shows that $A_1/2q^2 = y_0$. The special solution is therefore

$$y = y_0\left[1 + \sum \frac{2q^2 \cos nx}{q^2-n^2}\right] + \sum \frac{\sin nx}{q^2-n^2}.$$

This is absolutely and uniformly convergent for all x.

In 3.8 if q is replaced by a positive integer N (or zero) the solution fails owing to the zero denominator $q^2 - n^2$ for $n = N$. Supposing $N > 0$, the equation (a) then has the simple solution $y = A_1 \cos Nx + A_2 \sin Nx$, which is the complementary function for (b). To find a particular integral for (b) the procedure is to solve

3.9 $$(D^2 + N^2)\, y = \tfrac{1}{2}\cot\tfrac{1}{2}x$$

in a trigonometric series suitable for $(-\pi, \pi)$. As above, replace $\frac{1}{2}\cot\frac{1}{2}x$ by the series $\sum \sin nx$, substitute $y = \frac{1}{2}a_0 + \sum (a_n \cos nx + b_n \sin nx)$ and equate the difference of the two sides to the NTS $\Lambda_2(-\pi, \pi)$. Thus

$$N^2 \tfrac{1}{2}a_0 + \sum (N^2 - n^2)(a_n \cos nx + b_n \sin nx) - \sum \sin nx$$
$$= A_1 [\tfrac{1}{2} + \sum (-1)^n \cos nx] + A_2 \sum (-1)^n n \sin nx.$$

Equating coefficients gives

$$0 \times a_N = (-1)^N A_1, \quad 0 \times b_N = 1 + (-1)^N N A_2,$$

and, for $n \neq N$,

$$(N^2 - n^2)\, a_n = (-1)^n A_1, \quad (N^2 - n^2)\, b_n = 1 + (-1)^n n A_2,$$

including $N^2 a_0 = A_1$. Hence $A_1 = 0$ and $a_N = A'_1$ (an arbitrary constant); $A_2 = (-1)^{N+1}/N$ and $b_N = A'_2$ (an arbitrary constant); $a_0 = A_1/N^2 = 0$; and, for $n \neq N$,

$$a_n = \frac{(-1)^n A_1}{N^2 - n^2} = 0, \quad b_n = \frac{1 - (-1)^n n A_2}{N^2 - n^2} = \frac{N - (-1)^{n+N} n}{N(N^2 - n^2)}.$$

The complete solution of 3.9 is therefore

$$y = A'_1 \cos Nx + A'_2 \sin Nx + \sum{}' \frac{N - (-1)^{n+N} n}{N(N^2 - n^2)} \sin nx,$$

where $\sum'$ denotes summation with $n = N$ omitted. It may be observed that in such a case the complementary function is obtained incidentally in the process of finding the particular integral.

The case $N = 0$ is left to the reader; the solution is

$$y = A_1 + A_2 \sum \frac{(-1)^n \sin nx}{n} - \sum \frac{\sin nx}{n^2}.$$

4. Linear equation with constant coefficients. Applied to the general linear ordinary differential equation with constant coefficients the method of §3, using the NTS $\Lambda_m(-\pi, \pi)$, yields the following theorem.

THEOREM. *Let*

(i) *$F(D)$ be a polynomial of degree m in D with constant coefficients;*

(ii) *$f(x) = \sum \gamma_n e^{inx}$,* (C, k), *where k is a non-negative integer, uniformly over $(-\pi, \pi)$, with the possible exclusion of the neighbourhoods of a finite number of exceptional points.*

Then the general solution of the ordinary differential equation

4.1 $$F(D)\, y = f(x)$$

is representable in $(-\pi, \pi)$ by a trigonometric series

4.2 $$y = \sum c_n e^{inx}$$

which is convergent if $k < m$ and summable (C, $k - m$) *if $k > m$, uniformly as in* (ii).

The coefficients c_n for which $F(in) \neq 0$ are given by

4.3 $$c_n = \left[\gamma_n + (-1)^n \sum_{r=1}^{m} A_r n^{r-1}\right] / F(in).$$

The coefficients c_N, if any, for which $F(iN) = 0$ are arbitrary constants and, for every such N,

4.4 $$\gamma_N + (-1)^N \sum_{r=1}^{m} A_r N^{r-1} = 0.$$

The constants A_r, with the c_N, form a set of m arbitrary constants.

Proof. First, by expanding $1/F(in)$ in powers of $1/n$ and appealing to the theorem on convergence factors referred to in §3 [**2**, p. 131, Theorem 76], it is seen from 4.2, 4.3 that the series for y is convergent if $k < m$ and summable (C, $k - m$) if $k > m$, uniformly as required.

Secondly, from 4.2, 4.3, 4.4, we have formally

$$
\begin{aligned}
F(D)y &= \sum{}' \gamma_n e^{inx} + \sum_{r=1}^{m} A_r \sum{}' (-1)^n n^{r-1} e^{inx} + \sum_N c_N F(iN) e^{iNx} \\
&= \sum \gamma_n e^{inx} - \sum_N \left[\gamma_N + (-1)^N \sum_{r=1}^{m} A_r N^{r-1} \right] e^{iNx} + \sum_{r=1}^{m} A_r \sum (-1)^n n^{r-1} e^{inx} \\
&= f(x),
\end{aligned}
$$

where $\sum_N$ denotes summation over all integral values of N for which $F(iN) = 0$, $\sum'$ denotes summation over all other integral values of n and the summability is (C, k) or (C, m) according as $k > m$ or not.

Finally the justification of these formal equalities follows by observing that $F(D)$ is a linear combination of powers D^{r-1} with $r = 1, 2, \dots, m+1$ and that the conditions of the theorem on differentiation of summable series referred to in §3 [**2**, p. 349, Theorem 249) are satisfied in the closed intervals excluding the exceptional neighbourhoods. Therefore the various derived series are appropriately summable to the corresponding derivatives of y; and their combination, which is summable to $f(x)$, is summable to $F(D)y$. The theorem is established.

It may be observed that no knowledge of the factors of $F(D)$ is needed beyond that of whether or not $D - iN$ is a factor (i.e., $F(iN) = 0$) for integral N. If the coefficients of $F(D)$ are given algebraically, the existence and values of the integers N may or may not be determinable, but no difficulty can arise with numerical coefficients.

It may also be noted that the class of functions $f(x)$ which are representable as Cesàro sums of trigonometric series as required in the theorem includes in particular (a) functions which may have in $(-\pi, \pi)$ a finite number of discontinuities (or of "discontinuities in the mean" in the Cesàro sense [**3**]) and (b) functions which, regarded as analytic functions,[3] are meromorphic in a region including $(-\pi, \pi)$.

It is observed by the referee that the complementary function of 4.1 can be found independently by finding the Fourier series for the C.F. in the usual form involving the unknown linear factors of $F(D)$ and combining the resulting series.

Examples of equation of this type (besides those of §3) are

(a) $(D^3 + D^2 + 1)\, y = \frac{1}{2} \cot \frac{1}{2}x$, (b) $(D^2 + 1)(D^3 + D + 1)\, y = x^{-1}$,

whose solutions are

---

[3]Such functions can be reduced to the special function $\frac{1}{2} \cot \frac{1}{2}x$ and its derivatives.

(a) $$A_1[\tfrac{1}{2} + \sum(-1)^n\{(1 - n^2)\cos nx - n^3 \sin nx\}/\Delta$$
$$+ A_2\sum(-1)^n\{n^4 \cos nx + (n - n^3)\sin nx\}/\Delta$$
$$+ A_3\sum(-1)^n\{(n^2 - n^4)\cos nx - n^5 \sin nx\}/\Delta$$
$$- \sum\{n^3 \cos nx + (1 - n^2)\sin nx\}/\Delta,$$

with $\Delta = (1 - n^2)^2 + n^6$;

(b) $$\sum{}' 2\,\mathrm{Si}(n\pi)\{(n^3 - n)\cos nx + \sin nx\}/\pi R$$
$$+ A_1 \cos x + A_2 \sin x + A_3[\tfrac{1}{2} + \sum{}' \theta\{\cos nx - (n^3 - n)\sin nx\}]$$
$$+A_4\sum{}' \theta\{(n^4 - n^2)\cos nx + n \sin nx\} + A_5\sum{}' \theta\{n^2 \cos nx - (n^5 - n^3)\sin nx\}$$
$$+A_6\sum{}' \theta\{(n^6 - n^4)\cos nx + n^3 \sin nx\} + A_7\sum{}' \theta\{n^4 \cos nx - (n^7 - n^5)\sin nx\},$$

with $\theta^{-1} = (-1)^n R = (-1)^n(1 - n^2)\{1 + (n - n^3)^2\}$, $A_3 + A_5 + A_7 = 0$, $A_4 + A_6 = 2\,\mathrm{Si}\,\pi/\pi$; where $\sum$ and $\sum'$ denote summation for n from 1 to ∞ and from 2 to ∞ respectively, and $\mathrm{Si}(x)$ denotes

$$\int_0^x \frac{\sin t}{t}\,dt.$$

5. Concluding remarks. Although the principles of the method, as described in §3, are not restricted to ordinary linear equations with constant coefficients, the algebraic difficulties involved in an attempt to apply it even to linear equations with non-constant coefficients would seem forbidding. Extension to partial equations in two or more variables, using such null series as

$$\sum \exp(imx + iny)$$

might be more hopeful.

I am indebted to the referee for useful criticism.

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GENERAL TRANSFORMATIONS OF BILATERAL COGNATE TRIGONOMETRICAL SERIES OF ORDINARY HYPERGEOMETRIC TYPE

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1. Introduction. Whipple [6] was the first to consider transformations connecting well-poised hypergeometric series as particular cases of relations between cognate trigonometrical series. He used contour integrals of the Barnes type to deduce such transformations. Later Sears [3] gave a systematic theory of general and well-poised transformations of trigonometrical series of any order which included Whipple's results as particular cases.

The object of this paper is to study general transformations of bilateral cognate trigonometrical series in analogy with the ordinary bilateral series introduced by Bailey [1].

Thus, in §2 bilateral trigonometrical series have been defined and later, with the help of Sear's known transformations of ordinary trigonometrical series, a number of transformations connecting general and well-poised series have been deduced in §§4–9.

In §10 particular cases of these transformations have been considered which yield the known transformations of Bailey [1], Slater [4; 5], Jackson [2], and Sears [3].

Lastly, in §11 and onwards I have given direct proofs of the bilateral transformations deduced in §§4–9 by employing contour integrals of the Barnes type similar to those employed by Slater [5] for deducing Sears' general transformations of generalized hypergeometric series.

2. Definitions. If u_n denotes the $(n+1)$th term of the series

$$F(a_1, a_2, \dots, a_{M+1};\ b_1, b_2, \dots, b_M;)$$

then, following Sears' notation, the series

$$\sum_n (-)^n u_n \sin(\lambda + 2n)\theta, \qquad \sum_n u_n \sin(\lambda + 2n)\theta,$$

$$\sum_n (-)^n u_n \cos(\lambda + 2n)\theta, \qquad \sum_n u_n \cos(\lambda + 2n)\theta$$

will be denoted by the symbols

$$_{M+1}S_M\begin{bmatrix} a_1, \dots, a_{M+1};\ \theta \\ b_1, \dots, b_M\ \ ;\ \lambda \end{bmatrix}, \qquad _{M+1}S'_M\begin{bmatrix} a_1, \dots, a_{M+1};\ \theta \\ b_1, \dots, b_M\ \ ;\ \lambda \end{bmatrix}$$

$$_{M+1}C_M\begin{bmatrix} a_1, \dots, a_{M+1};\ \theta \\ b_1, \dots, b_M\ \ ;\ \lambda \end{bmatrix}, \qquad _{M+1}C'_M\begin{bmatrix} a_1, \dots, a_{M+1};\ \theta \\ b_1, \dots, b_M\ \ ;\ \lambda \end{bmatrix}$$

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respectively. When $\lambda = a_1$, the first numerator parameter, it will be omitted from each symbol, and, following Whipple, series of the type

$$S\begin{bmatrix} a_1, \ldots, a_{M+1}; \theta \\ 1 + a_1 - a_2, \ldots, 1 + a_1 - a_{M+1}, \end{bmatrix}$$

in which $\lambda = a_1$, necessarily, will be called well-poised and will be denoted by ${}_{M+1}S_M(a_1)$. We will also denote series of the type

$$S\begin{bmatrix} 2a_r - a_1, a_r, a_r + a_2 - a_1, \ldots, a_r + a_{M+1} - a_1; \theta \\ 1 + a_r - a_1, \ldots, 1 + a_r - a_{M+1}, \end{bmatrix}$$

by the symbol ${}_{M+1}S_M(a_r)$ for $r > 1$; where the numerator parameter $(a_r + a_r - a_1)$ is the first to occur, and the denominator parameter $(1 + a_r - a_r)$ is omitted. Similar notations are used for the series C, S', and C'.

Now we define the bilateral series as follows:

$$\begin{matrix}(2.1)\\(2.2)\end{matrix}\qquad \begin{matrix}{}_pX_p\\{}_pZ_p\end{matrix}\begin{bmatrix} a_1, \ldots, a_p; \theta \\ b_1, \ldots, b_p; \lambda \end{bmatrix} = \sum_{-\infty}^{\infty} \frac{(-)^n (a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_p)_n} \begin{matrix}\sin\\\cos\end{matrix}\left\{(\lambda + 2n)\theta\right\}$$

$$\begin{matrix}(2.3)\\(2.4)\end{matrix}\qquad \begin{matrix}{}_pX'_p\\{}_pZ'_p\end{matrix}\begin{bmatrix} a_1, \ldots, a_p; \theta \\ b_1, \ldots, b_p; \lambda \end{bmatrix} = \sum_{-\infty}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_p)_n} \begin{matrix}\sin\\\cos\end{matrix}\left\{(\lambda + 2n)\theta\right\}$$

where $(\alpha)_n = \Gamma(\alpha + n)/\Gamma(\alpha)$.

It is easily seen that

$$\begin{matrix}(2.5)\\(2.6)\end{matrix}\qquad \begin{matrix}{}_pX_p\\{}_pZ_p\end{matrix}\begin{bmatrix} a_1, \ldots, a_p; \theta \\ b_1, \ldots, b_p; \lambda \end{bmatrix} = \begin{matrix}{}_{p+1}S_p\\{}_{p+1}C_p\end{matrix}\begin{bmatrix} 1, a_1, \ldots, a_p; \theta \\ b_1, \ldots, b_p; \lambda \end{bmatrix}$$

$$\pm \frac{(1 - b_1) \ldots (1 - b_p)}{(1 - a_1) \ldots (1 - a_p)} \begin{matrix}{}_{p+1}S_p\\{}_{p+1}C_p\end{matrix}\begin{bmatrix} 1, 2 - b_1, \ldots, 2 - b_p; \theta \\ 2 - a_1, \ldots, 2 - a_p;\ 2 - \lambda \end{bmatrix},$$

where on the right-hand side the positive or negative sign is to be taken according as the series is of the type ${}_pX_p$ or ${}_pZ_p$.

Similarly,

$$\begin{matrix}(2.7)\\(2.8)\end{matrix}\qquad \begin{matrix}{}_pX'_p\\{}_pZ'_p\end{matrix}\begin{bmatrix} a_1, \ldots, a_p; \theta \\ b_1, \ldots, b_p; \lambda \end{bmatrix} = \begin{matrix}{}_{p+1}S'_p\\{}_{p+1}C'_p\end{matrix}\begin{bmatrix} 1, a_1, \ldots, a_p; \theta \\ b_1, \ldots, b_p; \lambda \end{bmatrix}$$

$$\mp \frac{(1 - b_1) \ldots (1 - b_p)}{(1 - a_1) \ldots (1 - a_p)} \begin{matrix}{}_{p+1}S'_p\\{}_{p+1}C'_p\end{matrix}\begin{bmatrix} 1, 2 - b_1, \ldots, 2 - b_p; \theta \\ 2 - a_1, \ldots, 2 - a_p; 2 - \lambda \end{bmatrix},$$

where on the right-hand side the negative or positive sign is to be taken according as the series is of the type ${}_pX'_p$ or ${}_pZ'_p$.

The convergence factor $\Re(\Sigma b_p - \Sigma a_p - 1)$ will be henceforth denoted by y.

The series (2.1) and (2.2) converge when either

$$y > 0, \quad -\pi < 2\theta < \pi$$

or

$$-1 < y < 0, \quad -\pi < 2\theta < \pi.$$

The series (2.3) and (2.4) converge when either

$$y > 0, \quad 0 < \theta < \pi$$

or

$$-1 < y < 0, \quad 0 < \theta < \pi.$$

When $y > 0$ all the series converge uniformly and absolutely in the variable θ or in λ but when $-1 < y < 0$ the convergence is, in general, conditional.

3. Notation. Let

$$G(a_1, \ldots, a_M; b_1, \ldots, b_N) = \left\{\prod_{r=1}^{M} \Gamma(a_r)\right\}\left\{\prod_{1}^{N} \Gamma(b_r)\right\}^{-1}$$

$$A = \left\{\prod_{2}^{M} G(a_r; b_r)\right\}\left\{\prod_{M+1}^{M+N} G(1 - b_r; 1 - a_r)\right\}$$

$$A(a_1) = \left\{\prod_{2}^{M} G(a_r - a_1; b_r - a_1)\right\}\left\{\prod_{M+1}^{M+N} G(1 + a_1 - b_r; 1 + a_1 - a_r)\right\} \times \left\{\Gamma(b_1 - a_1)\right\}^{-1}$$

$$A(b_{M+1}) = \left\{\prod_{1}^{M} G(1 + a_r - b_{M+1}; 1 + b_r - b_{M+1})\right\} \times \left\{\prod_{M+2}^{M+N} G(b_{M+1} - b_r; b_{M+1} - a_r)\right\}^{-1} \left\{\Gamma(b_{M+1} - a_{M+1})\right\}^{-1}$$

$$B = A \operatorname{cosec} \pi a_{M+1}/\Gamma(1 - a_{M+N+1})$$

$$B(a_1) = A(a_1)\, G(a_1; 1 + a_1 - a_{M+N+1}) \operatorname{cosec} \pi(a_{M+1} - a_1)$$

$$B(b_{M+1}) = -A(b_{M+1})\, G(b_{M+1} - 1; b_{M+1} - a_{M+N+1}) \operatorname{cosec} \pi(b_{M+1} - a_{M+1})$$

$$P = \Gamma(a_1)\left\{\prod_{2}^{M+1} G(a_r, a_r - a_1)\right\}\left\{\prod_{M+2}^{2M} G(1 + a_1 - a_r, 1 - a_r)\right\}^{-1}$$

$$Q(a_2) = \left\{\prod_{1}^{M+1}{}' G(a_r - a_2, a_2 + a_r - a_1)\right\} \times \left\{\prod_{M+2}^{2M} G(1 + a_2 - a_r, 1 + a_1 - a_2 - a_r)\right\}^{-1}$$

$$R = P/G(1 + a_1 - a_{2M+1}, 1 - a_{2M+1})$$

$$T(a_2) = Q(a_2)/G(1 + a_2 - a_{2M+1}, 1 + a_1 - a_2 - a_{2M+1})$$

$$U = \Gamma(a_1)\left\{\prod_{2}^{M+2} G(a_r, a_r - a_1)\right\}\left\{\prod_{M+3}^{2M+1} G(1 + a_1 - a_r, 1 - a_r)\right\}^{-1}$$

$$V(a_2) = \left\{\prod_{1}^{M+2}{}' G(a_r - a_2, a_r + a_2 - a_1)\right\} \times \left\{\prod_{M+3}^{2M+1} G(1 + a_2 - a_r, 1 + a_1 - a_2 - a_r)\right\}^{-1}$$

$$X = U\,G(1 + a_1 - a_{2M+1}, 1 - a_{2M+1})$$

$$Y(a_2) = V(a_2)\,G(1 + a_2 - a_{2M+1}, 1 + a_1 - a_2 - a_{2M+1}).$$

The primes in the product symbols denote the omission of the gamma functions with zero arguments.

4. The general bilateral transformations. Sears [3] has proved the following general theorem for ordinary cognate trigonometrical series:

$$B\begin{matrix}S\\C\end{matrix}\left[\begin{matrix}a_1, \ldots, a_{M+N+1}; & \theta\\ b_1, \ldots, b_{M+N} & ; \lambda\end{matrix}\right] \tag{4.1, 4.2}$$

$$= \mp B(a_1)\begin{matrix}S\\C\end{matrix}\left[\begin{matrix}a_1, 1 + a_1 - b_1, \ldots, 1 + a_1 - b_{M+N} & ; & \theta\\ 1 + a_1 - a_2, \ldots, 1 + a_1 - a_{M+N+1}; & - & \lambda + 2a_1(1 - m\pi/\theta)\end{matrix}\right]$$

$$\mp \text{ idem } (a_1; a_2, \ldots, a_{M+1})$$

$$+ B(b_{M+1})\begin{matrix}S\\C\end{matrix}\left[\begin{matrix}1 + a_1 - b_{M+1}, \ldots, 1 + a_{M+N+1} - b_{M+1} & ;\theta\\ 2 - b_{M+1}, 1 + b_1 - b_{M+1}, \ldots, 1 + b_{M+N} - b_{M+1}; & \\ & \lambda + 2 - 2b_{M+1}(1 - m\pi/\theta)\end{matrix}\right]$$

$$+ \text{ idem } (b_{M+1}; b_{M+2}, \ldots, b_{M+N}),$$

valid when either

$$y > 0, \quad (2m - 1)\pi < 2\theta < (2m + 1)\pi$$

or

$$-1 < y < 0, \quad (2m - 1)\pi < 2\theta < (2m + 1)\pi,$$

where m is a positive integer.

In (4.1) and (4.2) let us put $b_{M+1} = a_1, \ldots, b_{M+N} = a_M$ with $N = M$ and also let $a_{M+1} \to 1$. We find that all the series in (4.1) and (4.2) reduce to series of the type ${}_{M+1}S_M$ (or ${}_{M+1}C_M$) and the series corresponding to the parameter a_{M+1} on the right combines with the series on the left-hand side to give a bilateral series of the type ${}_MX_M$ (or ${}_MZ_M$). Also, the series corresponding to a_r combines with the series corresponding to b_{M+r} on the right for $r = 1, 2, \ldots, M$, to give corresponding bilateral series.

Finally, writing c_1 for a_{M+2}, c_2 for $a_{M+3}, \ldots, c_M$ for a_{2M+1}, we get the transformations

$$\prod_1^M G(a_r, 1 - a_r; b_r, 1 - c_r)\begin{matrix}{}_MX_M\\{}_MZ_M\end{matrix}\left[\begin{matrix}c_1, \ldots, c_M; \theta\\ b_1, \ldots, b_M; \lambda\end{matrix}\right] \tag{4.3, 4.4}$$

$$= \mp \Gamma(a_1)\Gamma(1 - a_1)\prod_1^{M}{}' G(a_r - a_1, 1 + a_1 - a_r; b_r - a_1, 1 + a_1 - c_r)$$

$$\times \begin{matrix}{}_MX_M\\{}_MZ_M\end{matrix}\left[\begin{matrix}1 + a_1 - b_1, \ldots, 1 + a_1 - b_M; & \theta\\ 1 + a_1 - c_1, \ldots, 1 + a_1 - c_M; & -\lambda + 2a_1(1 - m\pi/\theta)\end{matrix}\right]$$

$$\mp \text{ idem } (a_1; a_2, \ldots, a_M)$$

valid when either

$$y > 0, \quad (2m-1)\pi < 2\theta < (2m+1)\pi$$

or

$$-1 < y < 0, \quad (2m-1)\pi < 2\theta < (2m+1)\pi.$$

5. Sears has also proved that

(5.1)
(5.2)
$$B\,\begin{matrix} S' \\ C' \end{matrix}\begin{bmatrix} a_1, \ldots, a_{M+N+1}; & \theta \\ b_1, \ldots, b_{M+N} & ; \lambda \end{bmatrix}$$

$$= \mp B(a_1)\,\begin{matrix} S' \\ C' \end{matrix}\begin{bmatrix} a_1, 1+a_1-b_1, \ldots, 1+a_1-b_{M+N}; & \theta \\ 1+a_1-a_2, \ldots, 1+a_1-a_{M+N+1}; & -\lambda + 2a_1 - (2m+1)\pi a_1/\theta \end{bmatrix}$$

$$\mp \text{ idem } (a_1; a_2, \ldots, a_{M+1})$$

$$- B(b_{M+1})\,\begin{matrix} S' \\ C' \end{matrix}\begin{bmatrix} 1+a_1-b_{M+1}, \ldots, 1+a_{M+N+1}-b_{M+1} & ;\theta \\ 2-b_{M+1}, 1+b_1-b_{M+1}, \ldots, 1+b_{M+N}-b_{M+1}; & \lambda + 2(1-b_{M+1}) + (2m+1)\pi b_{M+1}/\theta \end{bmatrix}$$

$$- \text{ idem } (b_{M+1}; b_{M+2}, \ldots, b_{M+N}).$$

Using the same substitutions as in §4 we arrive at the following transformations

(5.3)
(5.4)
$$\prod_1^M G(a_r, 1-a_r; b_r, 1-c_r)\,\begin{matrix} {}_MX'_M \\ {}_MZ'_M \end{matrix}\begin{bmatrix} c_1, \ldots, c_M; \theta \\ b_1, \ldots, b_M; \lambda \end{bmatrix}$$

$$= \mp \Gamma(a_1)\Gamma(1-a_1)\prod_1^M G(a_r - a_1, 1+a_1-a_r; b_r - a_1, 1+a_1-c_r)$$

$$\times \begin{matrix} {}_MX'_M \\ {}_MZ'_M \end{matrix}\begin{bmatrix} 1+a_1-b_1, \ldots, 1+a_1-b_M; & \theta \\ 1+a_1-c_1, \ldots, 1+a_1-c_M; & -\lambda + 2a_1 - (2m+1)\pi a_1/\theta \end{bmatrix}$$

$$\mp \text{ idem } (a_1; a_2, \ldots, a_M)$$

valid under the conditions

$$y > 0, \quad m\pi < \theta < (m+1)\pi$$

or

$$-1 < y < 0, \quad m\pi < \theta < (m+1)\pi.$$

6. Well-poised trigonometrical transformations. Sears [3] has also proved the following nine transformation theorems for well-poised trigonometrical series.

(6.1) $P_{2M}\, S'_{2M-1}(a_1) + Q(a_2)\, {}_{2M}S'_{2M-1}(a_2) + \text{idem } (a_2; a_3, \ldots, a_{M+1}) = 0$

(6.2) $P \cos \tfrac{1}{2}\pi a_1\, {}_{2M}S_{2M-1}(a_1) + Q(a_2) \cos \tfrac{1}{2}\pi(2a_2 - a_1)\, {}_{2M}S_{2M-1}(a_2)$
$+ \text{idem } (a_2; a_3, \ldots, a_{M+1}) = 0$

(6.3) $P \sin \frac{1}{2}\pi a_1 \,{}_{2M}C_{2M-1}(a_1) + Q(a_2) \sin \frac{1}{2}\pi(2a_2 - a_1) \,{}_{2M}C_{2M-1}(a_2)$
$+ \text{idem } (a_2; a_3, \dots, a_{M+1}) = 0$

(6.4) $U \sin \pi a_1 \,{}_{2M+1}C_{2M}(a_1) + V(a_2) \sin \pi(2a_2 - a_1) \,{}_{2M+1}C_{2M}(a_2)$
$+ \text{idem } (a_2; a_3, \dots, a_{M+2}) = 0$

(6.5) $U \cos \frac{1}{2}\pi a_1 \,{}_{2M+1}S'_{2M}(a_1) + V(a_2) \cos \frac{1}{2}\pi(2a_2 - a_1) \,{}_{2M+1}S'_{2M}(a_2)$
$+ \text{idem } (a_2; a_3, \dots, a_{M+2}) = 0$

(6.6) $U \sin \frac{1}{2}\pi a_1 \,{}_{2M+1}C'_{2M}(a_1) + V(a_2) \sin \frac{1}{2}\pi(2a_2 - a_1) \,{}_{2M+1}C'_{2M}(a_2)$
$+ \text{idem } (a_2; a_3, \dots, a_{M+2}) = 0$

(6.7) $U \,{}_{2M+1}S_{2M}(a_1) + V(a_2) \,{}_{2M+1}S_{2M}(a_2) + \text{idem } (a_2; a_3, \dots, a_{M+2}) = 0$

(6.8) $X \sin \pi a_1 \,{}_{2M}C'_{2M-1}(a_1) + Y(a_2) \sin \pi(2a_2 - a_1) \,{}_{2M}C'_{2M-1}(a_2)$
$+ \text{idem } (a_2; a_3, \dots, a_{M+2}) = 0$

(6.9) $R \,{}_{2M+1}S_{2M}(a_1) + T(a_2) \,{}_{2M+1}S_{2M}(a_2) + \text{idem } (a_2; a_3, \dots, a_{M+1}) = 0$

The conditions of validity for (6.1), (6.5), (6.6), (6.8) are either $y > 0$, $0 < \theta < \pi$ or $-1 < y < 0$ and $0 < \theta < \pi$.

The results (6.2), (6.3), (6.4), and (6.9) are true for either $y > 0$, $-\pi < 2\theta < \pi$ or for $-1 < y < 0$ and $-\pi < 2\theta < \pi$.

The result (6.7), however, is true for either $y > 0$, $-3\pi < 2\theta < 3\pi$ or $-1 < y < 0$ and $-3\pi < 2\theta < 3\pi$ $(2\theta \neq \pm\pi)$.

7. Transformations of well-poised bilateral series. Consider first the transformations (6.1), (6.2), and (6.3). In each one of them put $M = 2N + 1$ (an odd integer) and then let $a_{2N+2} = 1$, $a_2 = 1 + a_1 - a_3$, $a_4 = 1 + a_1 - a_5$, etc., and, in general, $a_{2N} = 1 + a_1 - a_{2N+1}$. The series then reduce to one of the type ${}_{2N+1}S'_{2N}$, ${}_{2N+1}S_{2N}$, and ${}_{2N+1}C_{2N}$ respectively.

Simplifying the coefficients of the series on the left and, finally, writing a for a_1, b_1 for a_{2N+3}, b_2 for a_{2N+4}, etc. and, in general, b_{2N} for a_{4N+2}; a_1 for a_3, a_2 for a_5, etc. and in general, a_N for a_{2N+1}, we have, on combining the series in pairs as in §4, the following three transformations for ${}_{2N}X'_{2N}$, ${}_{2N}X_{2N}$, and ${}_{2N}Z_{2N}$ series, respectively:

(7.1) $$P' \,{}_{2N}X'_{2N}(a) = Q'(a_1) \,{}_{2N}X'_{2N}(a_1) + \text{idem } (a_1; a_2, \dots, a_N),$$

(7.2) $P' \cos \frac{1}{2}\pi a \,{}_{2N}X_{2N}(a) = Q'(a_1) \cos \frac{1}{2}\pi(2a_1 - a) \,{}_{2N}X_{2N}(a_1)$
$+ \text{idem } (a_1; a_2, \dots, a_N),$

(7.3) $P' \sin \frac{1}{2}\pi a \,{}_{2N}Z_{2N}(a) = Q'(a_1) \sin \frac{1}{2}\pi(2a_1 - a) \,{}_{2N}Z_{2N}(a_1)$
$+ \text{idem } (a_1; a_2, \dots, a_N),$

where

$$P' = \Gamma(a)\Gamma(1-a)\left\{\prod_1^N G(a_r, 1 - a_r, 1 + a - a_r, a_r - a)\right\} \times \left\{\prod_1^{2N} G(1 + a - b_r, 1 - b_r)\right\}^{-1},$$

$$Q'(a_1) = \Gamma(a_1 - a)\Gamma(1 + a - a_1)\Gamma(a_1)\Gamma(1 - a_1)$$
$$\times \left\{ \prod_1^N{}' G(1 + a - a_1 - a_r, a_1 + a_r - a, a_r - a_1, 1 + a_1 - a_r) \right\}$$
$$\times \left\{ \prod_1^{2N} G(1 + a_1 - b_r, 1 + a - a_1 - b_r) \right\}^{-1},$$

and ${}_MX_M(a)$, ${}_MX_M(a_r)$ denote the series

$${}_MX_M\left[\begin{matrix} b_1, b_2, \ldots, b_M & ; \theta \\ 1 + a - b_1, \ldots, 1 + a - b_M ; a \end{matrix}\right],$$

and

$${}_MX_M\left[\begin{matrix} a_r + b_1 - a, \ldots, a_r + b_M - a; & \theta \\ 1 + a_r - b_1, \ldots, 1 + a_r - b_M; 2a_r - a \end{matrix}\right]$$

respectively, with similar notations for X', Z, and Z' series also.

The transformations (7.1), (7.2), and (7.3) are valid under the same conditions as those required for (6.1), (6.2), and (6.3) respectively.

8. We now pass on to the consideration of (6.4), (6.5), (6.6), and (6.7). Taking $M = 2N$ (an even integer) and treating them by a method similar to that of §7, we get the transformations:

(8.1) $$P' \Gamma(1 + a - b_{2N}) \Gamma(1 - b_{2N}) \sin \pi a \; {}_{2N-1}Z_{2N-1}(a)$$
$$= Q'(a_1) \Gamma(1 + a_1 - b_{2N}) \Gamma(1 + a - a_1 - b_{2N}) \sin \pi(2a_1 - a) \; {}_{2N-1}Z_{2N-1}(a_1)$$
$$+ \text{idem } (a_1; a_2, \ldots, a_N),$$

(8.2) $$P' \Gamma(1 + a - b_{2N}) \Gamma(1 - b_{2N}) \cos \tfrac{1}{2}\pi a \; {}_{2N-1}X'_{2N-1}(a)$$
$$= Q'(a_1) \Gamma(1 + a_1 - b_{2N}) \Gamma(1 + a - a_1 - b_{2N}) \cos \tfrac{1}{2}\pi(2a_1 - a) {}_{2N-1}X'_{2N-1}(a_1)$$
$$+ \text{idem } (a_1; a_2, \ldots, a_N),$$

(8.3) $$P' \Gamma(1 + a - b_{2N}) \Gamma(1 - b_{2N}) \sin \tfrac{1}{2}\pi a \; {}_{2N-1}Z'_{2N-1}(a)$$
$$= Q'(a_1) \Gamma(1 + a_1 - b_{2N}) \Gamma(1 + a - a_1 - b_{2N}) \sin \tfrac{1}{2}\pi(2a_1 - a) \; {}_{2N-1}Z'_{2N-1}(a_1)$$
$$+ \text{idem } (a_1; a_2, \ldots, a_N),$$

(8.4) $$P' \Gamma(1 + a - b_{2N}) \Gamma(1 - b_{2N}) \; {}_{2N-1}X_{2N-1}(a)$$
$$= Q'(a_1) \Gamma(1 + a_1 - b_{2N}) \Gamma(1 + a - a_1 - b_{2N}) \; {}_{2N-1}X_{2N-1}(a_1)$$
$$+ \text{idem } (a_1; a_2, \ldots, a_N),$$

respectively. These transformations are valid under the same conditions as those necessary for (6.4), (6.5), (6.6), and (6.7) respectively.

It may be noted that the transformations (8.1), (8.2), and (8.4) can also be obtained by putting $b_{2N} = \frac{1}{2}(1 + a)$ in (7.3), (7.1), and (7.2) respectively.

9. Next, in (6.8) take $M = 2N$ and, after reduction as in §7 and §9, change N to $(N + 1)$. This gives the transformation

(9.1) $$P'\Gamma(1 + a - a_{N+1})\Gamma(a_{N+1} - a)\Gamma(1 - a_{N+1})\Gamma(a_{N+1}) \sin \pi a \; {}_{2N}Z'_{2N}(a)$$
$$= Q'(a_1) \Gamma(a_1 - a + a_{N+1}) \Gamma(1 + a - a_1 - a_{N+1}) \Gamma(a_{N+1} - a_1)$$
$$\times \Gamma(1 + a_1 - a_{N+1}) \sin \pi(2a_1 - a) \; {}_{2N}Z'_{2N}(a_1) + \text{idem } (a_1; a_2, \ldots, a_{N+1}).$$

Lastly, in (6.9), take $M = 2N + 1$ and as before we obtain the transformation

$$(9.2)\quad \{P'/\Gamma(1 + a - b_{2N+1})\,\Gamma(1 - b_{2N+1})\}\ {}_{2N+1}X_{2N+1}(a)$$
$$= \{Q'(a_1)/\Gamma(1 + a_1 - b_{2N+1})\,\Gamma(1 + a - a_1 - b_{2N+1})\}\ {}_{2N+1}X_{2N+1}(a_1) + \text{idem}\,(a_1; a_2, \ldots, a_N).$$

10. Special cases. In this section we shall briefly mention some of the interesting particular cases of the transformations deduced in §§4–9.

If we take $\theta = 0$ or $\frac{1}{2}\pi$ in all the above transformations we get the transformations of ordinary bilateral series with arguments $+1$ or -1, as the case may be.

In particular, $\theta = \frac{1}{2}\pi$ in (7.1) gives Slater [**4**, (11)]. Also, $\theta = 0$ in (7.3), (8.1), (8.3), (9.1) gives Slater [**4**, (11), (14), (13), (12)].

As indicated by Slater [**4**] her trasnformations include all those of Bailey [**1**].

We can also by suitable substitutions rediscover Sears' original theorems on trigonometric series.

11. Bilateral trigonometrical integrals. We shall now give a direct proof of the general transformations deduced in §§4–9. Let us first consider the general transformation (4.3). For simplicity we shall prove it for the case when $m = 0$.

Consider the integral

$$\frac{1}{2\pi i}\int_C G\begin{bmatrix} a_1 + s, \ldots, a_M + s, 1 - a_1 - s, \ldots, 1 - a_M - s, -s, 1 + s; \\ b_1 + s, \ldots, b_M + s, 1 - c_1 - s, \ldots, 1 - c_M - s, \end{bmatrix} e^{2is\theta} ds$$

where the contour of integration C is a circle of radius R with origin as centre, and R is so chosen that the circle does not pass through any of the poles of the integrand. The parameters a, b, and c are supposed real for simplicity's sake. θ is necessarily a real quantity. The parameters are such that none of the members of the two sequences

$$-1 - n, -n - a_1, \ldots, -n - a_M$$

and

$$n, 1 - a_1 + n, \ldots, 1 - a_M + n$$

coincide.

Now the integrand for $\Re s > 0$ can be written as

$$\frac{\Gamma(c_1 + s) \ldots \Gamma(c_M + s)}{\Gamma(b_1 + s) \ldots \Gamma(b_M + s)}\ \frac{\pi \sin \pi(c_1 + s) \ldots \sin \pi(c_M + s)}{\sin \pi s \sin \pi(a_1 + s) \ldots \sin \pi(a_M + s)}\, e^{2is\theta}.$$

Writing $s = Re^{i\phi}$, $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$, we find that the first fraction in the above product is

$$O(R^{\Sigma c_r - \Sigma b_r})$$

and the second fraction is

(i) bounded for $R \to \infty$ if $-\pi < 2\theta < \pi$, and

(ii) $O[\exp(-2\theta R \sin\phi - R\pi|\sin\phi|)]$ when $R \to \infty$.

From (i) it follows that the integral round the semicircle on the right of the imaginary axis tends to zero if

$$\left(\sum_1^M b_r - \sum_1^M c_r\right) > 1, \qquad -\pi < 2\theta < \pi.$$

Again from (ii) it follows with the help of Jordan's Lemma [**7**, p. 115, §6.222] that the integral round the same semicircle tends to zero if

$$\left(\sum_1^M b_r - \sum_1^M c_r\right) > 0, \qquad -\pi < 2\theta < \pi.$$

Similarly, when $\Re s < 0$ we write the integrand as

$$\frac{\Gamma(1-b_1-s)\ldots\Gamma(1-b_M-s)}{\Gamma(1-c_1-s)\ldots\Gamma(1-c_M-s)} \frac{\pi \sin\pi(b_1+s)\ldots\sin\pi(b_M+s)}{\sin\pi s\sin\pi(a_1+s)\ldots\sin\pi(a_M+s)} e^{2is\theta}$$

and similar remarks follow under the same conditions. Thus we have shown that the integral round the circle $|z| = R$ tends to zero as $R \to \infty$ under the above two sets of conditions.

For the sake of brevity let us suppose that

$$r_{n,m} \qquad (m = 0, 1, \ldots, M)$$

are the residues of the integrand at the poles $n, 1 - a_1 + n, \ldots, 1 - a_M + n$, and

$$R_{n,m} \qquad (m = 0, 1, \ldots, M)$$

are the residues of the integrand at the poles $-1-n, -n-a_1, \ldots, -n-a_M$.

Then, if $\sum b_r - \sum c_r > 1$ and $-\pi < 2\theta < \pi$, the series of residues

$$\sum_{n=0}^{\infty} r_{n,m}, \quad \sum_{n=0}^{\infty} R_{n,m} \qquad (m = 0, 1, \ldots, M),$$

converge absolutely and there is no difficulty in letting $R \to \infty$. Hence, by Cauchy's Theorem,

$$\sum_{m=0}^{M}\sum_{n=0}^{\infty} r_{n,m} + \sum_{m=0}^{M}\sum_{n=0}^{\infty} R_{n,m} = 0.$$

Introducing a parameter λ by multiplying the above equation by $e^{i\lambda\theta}$ throughout and equating the real and imaginary parts we obtain, after combining the series $\sum r_{n,s}$ with $\sum R_{n,s}$, the transformations (4.4) and (4.3) respectively.

In the case when $\sum b_r - \sum c_r > 0$ and $-\pi < 2\theta < \pi$, the series of residues are only conditionally convergent, but, with a little labour we can justify the limiting process of letting $R \to \infty$. The details are similar to those given by Whipple [**6**] and hence have been omitted.

In the case when the parameters a, b, and c are complex, we have to replace the conditions of convergence by their real parts and instead of equating the real and imaginary parts in the final series of residues we have to add or subtract the conjugate series.

In order to obtain the transformations (5.3) and (5.4) we consider the above integral with $e^{i(\pi+2\theta)s}$ instead of the exponential factor $e^{2i\theta s}$ in the integrand, and proceed in exactly the same manner.

12. Well-poised bilateral trigonometrical integrals. If

$$
\begin{aligned}
P_N(s) = {} & \\
& \Gamma(a+s)\,\Gamma(a_1+s)\ldots\Gamma(a_N+s)\,\Gamma(1+a-a_1+s)\ldots\Gamma(1+a-a_N+s) \\
& \times \Gamma(a_1-a-s)\ldots\Gamma(a_N-a-s)\,\Gamma(-s)\,\Gamma(1+s)\,\Gamma(1-a-s) \\
& \qquad \times \Gamma(1-a_1-s)\ldots\Gamma(1-a_N-s)
\end{aligned}
$$

and

$$
Q_{2N}(s) = \left\{ \prod_{r=1}^{2N} G(1+a-b_r+s,\, 1-b_r-s) \right\}^{-1}
$$

then

$$
\frac{1}{2\pi i}\int_C P_N(s)\, Q_{2N}(s) \cos\left(\tfrac{1}{2}a+s\right)\pi\, e^{i\theta(2s+a)}\,ds, \tag{12.1}
$$

$$
\frac{1}{2\pi i}\int_C P_N(s)\, Q_{2N}(s)\, e^{i\theta(2s+a)}\,ds, \tag{12.2}
$$

$$
\frac{1}{2\pi i}\int_C P_N(s)\, Q_{2N}(s) \sin\left(\tfrac{1}{2}a+s\right)\pi\, e^{i\theta(2s+a)}\,ds, \tag{12.3}
$$

where C is the same contour as before, give the transformations (7.2), (7.1), (7.3), respectively. The rest of the transformations can easily be obtained by obvious modifications of the above integrals.

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FOURIER TRANSFORMS OF DISTRIBUTION FUNCTIONS

F. J. DYSON

A distribution function $\phi(x)$ is assumed to have the following properties:

$$\phi(x) \text{ is non-decreasing} \tag{1}$$

$$\lim_{x\to-\infty} \phi(x) = 0, \quad \lim_{x\to+\infty} \phi(x) = 1, \tag{2}$$

$$\phi(x) = \lim_{y\to x+0} \phi(y) \text{ for every } x. \tag{3}$$

The Fourier transform of $\phi(x)$ is defined by the Stieltjes integral

$$\Phi(t) = \int_{-\infty}^{\infty} e^{-itx}\, d\phi(x). \tag{4}$$

Let ϕ_1 and ϕ_2 be two distribution functions. Let a positive real number δ be given. We consider the question, does there exist a positive ϵ such that the condition

$$|\Phi_1(t) - \Phi_2(t)| < \epsilon \text{ for all } t \tag{5}$$

implies

$$|\phi_1(x) - \phi_2(x)| < \delta\ ? \tag{6}$$

There are three separate problems here. (i) We may allow ϵ to depend on δ, ϕ_1, and x. Then our question is, does the uniform convergence of Φ_2 to Φ_1 imply a point-wise convergence of ϕ_2 to ϕ_1?. The answer to this question is yes, as is well known; in fact Lévy [**1**, p. 49] proves a theorem which states considerably more than is needed for our problem. (ii) We may allow ϵ to depend on δ and ϕ_1, but not on x. Then our question is, does uniform convergence of Φ_2 to Φ_1 imply uniform convergence of ϕ_2 to ϕ_1? The answer to this question is also yes; we prove this in Theorem 1 below. (iii) We may allow ϵ to depend on δ only. In this case the answer is no, as we shall show by an example.

Counter-example for case (iii). Let a and b be real numbers with $b > a > 0$. We consider the distribution functions

$$\phi_1(x) = \begin{cases} \frac{1}{2}\log\left(\dfrac{x^2+b^2}{x^2+a^2}\right) \Big/ \log\left(\dfrac{b}{a}\right), & x < 0 \\ 1, & x > 0. \end{cases} \tag{7}$$

$$\phi_2(x) = 1 - \phi_1(-x). \tag{8}$$

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Then

$$\phi_1(x) - \phi_2(x) = \tfrac{1}{2}\log\left(\frac{x^2+b^2}{x^2+a^2}\right)\Big/\log\left(\frac{b}{a}\right), \qquad \text{all } x, \tag{9}$$

and in particular

$$\phi_1(0) - \phi_2(0) = 1. \tag{10}$$

However, by (9) we have

$$\Phi_1(t) - \Phi_2(t) = i\pi\frac{t}{|t|}[e^{-a|t|} - e^{-b|t|}]/\log\left(\frac{b}{a}\right), \tag{11}$$

$$|\Phi_1(t) - \Phi_2(t)| < \pi/\log\left(\frac{b}{a}\right). \tag{12}$$

Since b/a may be arbitrarily large, we see that we can satisfy (5) for any $\epsilon > 0$ and still have (6) false for $\delta = 1$.

Statement of theorem for case (ii).

THEOREM 1. *Let a positive δ and a distribution function ϕ_1 be given. Then we can find $\epsilon > 0$, depending only on δ and ϕ_1, such that* (5) *implies* (6) *for all x and for all ϕ_2.*

Let $h_\eta(x)$ be the function defined by

$$h_\eta(x) = \max(0, 1 - |x/\eta|). \tag{13}$$

Then (4) gives

$$\int_{-\infty}^{\infty} h_\eta(x-w)\,d\phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4\sin^2\frac{1}{2}\eta t}{\eta t^2}e^{itw}\,\Phi(t)\,dt, \tag{14}$$

both sides being absolutely convergent integrals. If ϵ is chosen so that (5) is satisfied, then (14) gives, for every η and w,

$$\left|\int_{-\infty}^{\infty} h_\eta(x-w)[d\phi_1(x) - d\phi_2(x)]\right| < \epsilon. \tag{15}$$

Since ϕ_1 is non-decreasing and (3) holds,

$$\phi_1(w) - \lim_{y\to w-0}\phi_1(y) = \lim_{\eta\to 0}\int h_\eta(x-w)\,d\phi_1(x), \tag{16}$$

the limits on both sides necessarily existing. Similarly (16) holds for ϕ_2. Therefore letting $\eta \to 0$ in (15), we have, for all w,

$$|(\phi_1(w) - \lim_{y\to w-0}\phi_1(y)) - (\phi_2(w) - \lim_{y\to w-0}\phi_2(y))| < \epsilon. \tag{17}$$

That is to say, at every point the discontinuities in ϕ_1 and ϕ_2 differ by at most ϵ. Another consequence of (15) is obtained by writing in turn $w+\eta, w+2\eta, \ldots, w+N\eta$ for w and adding the resulting inequalities. From the definition of $h_\eta(x)$,

$$\sum_{m=1}^{N} h_\eta(x - w - m\eta) = 1, \qquad w+\eta \le x \le w+N\eta,$$

and

$$0 < \sum_{m=1}^{N} h_\eta(x - w - m\eta) < 1,$$

$$w < x < w + \eta \text{ and } w + N\eta < x < w + (N+1)\eta.$$

Using the fact that ϕ_1 and ϕ_2 are non-decreasing, adding together (15) for these N values of w therefore gives

$$\int_{w}^{w+(N+1)\eta} d\phi_2(x) > \int_{w+\eta}^{w+N\eta} d\phi_1(x) - N\epsilon. \tag{18}$$

We write for brevity $\alpha = \frac{1}{4}\delta$. We can divide the whole line $(-\infty, +\infty)$ into a finite set of intervals $I_1, \ldots, I_m$ with the following properties. (i) Each I_n is closed on the left and open on the right. (ii) The total variation of $\phi_1(x)$ on I_n is less than α. Let L_n^1 and R_n^1 be the limits to which $\phi_1(x)$ tends as x tends to the left and right end-points within I_n. Similarly let L_n^2 and R_n^2 be the limits of ϕ_2. By (17) we have

$$R_n^2 - R_n^1 < L_{n+1}^2 - L_{n+1}^1 + \epsilon. \tag{19}$$

Now let λ be the length of the shortest I_n, let Λ be the combined length of $I_2, \ldots, I_{m-1}$, and let N be an integer greater than $(2\Lambda/\lambda)$. The choice of N and of the I_n depends only on δ and ϕ_1 and is independent of ϵ. Given any I_n with $1 < n < m$, we can choose two points x, x' inside I_n such that

$$x' - x > \tfrac{1}{2}\lambda. \tag{20}$$

Then we apply (18) with $w = x$, $w + \eta = x'$, giving

$$\phi_1(x') + \phi_2(x' + N\eta) > \phi_2(x) + \phi_1(x + N\eta) - N\epsilon. \tag{21}$$

By the definition of N, the point $(x + N\eta)$ belongs to I_m and so

$$\phi_1(x + N\eta) > 1 - \alpha, \quad \phi_2(x' + N\eta) < 1.$$

Hence (21) becomes

$$\phi_1(x') > \phi_2(x) - N\epsilon - \alpha. \tag{22}$$

Again, applying (18) with $w = x - N\eta$, $w + \eta = x' - N\eta$,

$$\phi_2(x') + \phi_1(x' - N\eta) > \phi_1(x) + \phi_2(x - N\eta) - N\epsilon,$$

and since $(x' - N\eta)$ belongs to I_1 this becomes

$$\phi_2(x') > \phi_1(x) - N\epsilon - \alpha. \tag{23}$$

Let x' and x tend respectively to the right and left to the end-points of I_n. Then (22) and (23) give

$$L_n^2 < R_n^1 + N\epsilon + \alpha, \tag{24}$$

$$R_n^2 > L_n^1 - N\epsilon - \alpha. \tag{25}$$

These inequalities, (24) and (25), which have been proved for $1 < n < m$, are trivially true also for $n = 1$ and $n = m$.

Writing $n + 1$ for n in (24) and combining it with (19), we find

$$\begin{aligned} R_n^2 &< R_n^1 + R_{n+1}^1 - L_{n+1}^1 + (N+1)\,\epsilon + \alpha \\ &< R_n^1 + (N+1)\,\epsilon + 2\alpha. \end{aligned} \tag{26}$$

Similarly (25) combined with (19) gives

$$L_n^2 > L_n^1 - (N+1)\,\epsilon - 2\alpha. \tag{27}$$

Now R_n^2 and L_n^2 are the upper and lower bounds of ϕ_2 in I_n, and R_n^1 and L_n^1 differ by at most α. Therefore (26) and (27) imply

$$\left|\phi_2(x) - \phi_1(x)\right| < (N+1)\epsilon + 3\alpha = (N+1)\epsilon + \tfrac{3}{4}\delta \tag{28}$$

for all x in $(-\infty, +\infty)$. The choice of N depended only on δ and ϕ_1. Given δ and ϕ_1 we can choose ϵ to be any number less than $(\delta/(4(N+1)))$, and then (5) will imply (6). This proves the theorem.

Additional remarks. Another theorem can be derived from Theorem 1 by weakening both the hypothesis and the conclusion slightly. Let us define the distance between two distributions ϕ_1 and ϕ_2 by

$$\left\|\phi_1 - \phi_2\right\| = \max\left(\left|\{\phi_1, \phi_2\}\right|, \left|\{\phi_2, \phi_1\}\right|\right), \tag{29}$$

where

$$\{\phi_1, \phi_2\} = \max_{x,x'}\left(\min\left(x' - x,\ \phi_1(x) - \phi_2(x')\right)\right). \tag{30}$$

This definition of the distance is equivalent to that given by Lévy [**1**, p. 47]. It is easy to see that $\|\phi_1 - \phi_2\|$ is the side of the largest square that can be inserted between the graphs $y = \phi_1(x)$ and $y = \phi_2(x)$ when these are plotted in cartesian coordinates in the usual way. Thus the convergence defined by $\|\phi_2 - \phi_1\| \to 0$ is topologically weaker than uniform convergence of ϕ_2 to ϕ_1, but topologically stronger than point-wise convergence of ϕ_2 to ϕ_1. The modified form of Theorem 1 is

THEOREM 2. *Let δ and ϕ_1 be given. Then we can find $\epsilon > 0$ depending only on δ and ϕ_1, such that*

$$\left|\Phi_1(t) - \Phi_2(t)\right| < \epsilon \textit{ for all } t < \frac{1}{\epsilon} \tag{31}$$

implies

$$\left\|\phi_2 - \phi_1\right\| < \delta. \tag{32}$$

The proof is similar to the proof of Theorem 1, only simpler. The counter-example given previously also shows that the weaker conclusion (32) does not follow from (5) with ϵ depending only on δ.

The author is indebted to Dr. K. L. Chung for suggesting this problem to him, and for several stimulating discussions.

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LINEAR COMBINATIONS OF BERNSTEIN POLYNOMIALS

P. L. BUTZER

1. Introduction. If $f(x)$ is defined on $[0, 1]$, then its corresponding Bernstein polynomial

$$(1)\quad B_n(x) = B_n^f(x) = \sum_{\nu=0}^{n} f(\nu n^{-1})\, p_{\nu,n}(x), \qquad p_{\nu,n}(x) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu},$$

approaches $f(x)$ uniformly on $[0, 1]$, if $f(x)$ is continuous on $[0, 1]$. If $f(x)$ is bounded on $[0, 1]$, then at every point x where the second derivative $f''(x)$ exists (Voronowskaja [**7**], see also [**5**])

$$\lim_{n\to\infty} n[B_n^f(x) - f(x)] = \frac{x(1-x)}{2} f''(x),$$

hence if $f''(x)$ is not zero on $[0, 1]$, the order of approximation to $f(x)$ by the $B_n(x)$ is exactly $O(n^{-1})$. It follows that the existence of derivatives of higher order of $f(x)$ cannot improve this order of approximation.

In this paper we shall introduce certain linear combinations of Bernstein polynomials which, under definite conditions, approximate $f(x)$ more closely than the Bernstein polynomials.

Polynomials approaching $f(x)$ more closely than the Bernstein polynomials, but of a different type from those considered here, were also considered by Bernstein [**1**] namely,

$$Q_n^f(x) = \sum_{\nu=0}^{n} \left[f(\nu n^{-1}) - \frac{x(1-x)}{2n} f''(\nu n^{-1}) \right] p_{\nu,n}(x).$$

Then if $|f(x)| < M$ and if $f^{(4)}(x)$ exists at the point x, it can be shown that

$$\lim_{n\to\infty} n^2[Q_n^f(x) - f(x)] = \frac{x(1-x)(1-2x)}{6} f^{(3)}(x) - \frac{[x(1-x)]^2}{8} f^{(4)}(x).$$

We remark that the combinations we consider do not contain the values of the derivatives of $f(x)$.

2. Preliminary results. We shall here recall some known facts, for their proofs one may consult [**5**, §§1.5–1.6]. With Bernstein [**1**] we define

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$$(2) \qquad S_{n,r}(x) = \sum_{\nu=0}^{n} (\nu n^{-1} - x)^r p_{\nu,n}(x) \qquad (n = 1, 2, \ldots; r = 0, 1, 2, \ldots)$$

and for $n^r S_{n,r}(x)$ we shall often write $T_{n,r}(x)$. If $f(x)$ is defined on [0, 1] with $|f(x)| < M$ then at points where $f^{(2k)}(x)$ exists [1],

$$(3) \qquad B_n^f(x) = f(x) + \sum_{r=1}^{2k} \frac{f^{(r)}(x)}{r!} S_{n,r}(x) + \frac{\epsilon_n}{n^k}$$

where $\epsilon_n \to 0$ as $n \to \infty$.

The recursion formula

$$T_{n,r+1}(x) = x(1-x)[T'_{n,r}(x) + nrT_{n,r-1}(x)]$$

is known, and by induction we obtain, putting $x(1-x) = X$,

$$T_{n,0} = 1, \quad T_{n,1} = 0, \quad T_{n,2} = nX, \quad T_{n,3} = n(1-2x)X,$$

$$(4) \quad T_{n,4} = 3n^2X^2 + n(X - 6X^2), \quad T_{n,5} = (1-2x)[10n^2X^2 + n(X - 12X^2)],$$

$$T_{n,6} = 15n^3X^3 + 5n^2X^2(5 - 26X) + nX[1 - 30X + 120X^2],$$

$$\cdots \qquad \cdots$$

In general, for r fixed, every $0 < x < 1$, $T_{n,r}(x)$ can be written as a polynomial in n,

$$(5) \quad T_{n,r}(x) = \phi_{r,r'}(x)\, n^{r'} + \phi_{r,r'-1}(x)\, n^{r'-1} + \phi_{r,r'-2}(x)\, n^{r'-2} + \ldots + \phi_{r,1}(x) n$$

of degree

$$r' \equiv [\tfrac{1}{2}r] = \begin{cases} \tfrac{1}{2}r & \text{for even } r, \\ \tfrac{1}{2}(r-1) & \text{for odd } r, \end{cases}$$

where the $\phi_{r,r'-i}(x)$ are polynomials in x, independent of n.

Moreover, for every r, one can show [5] there exists a constant A_r (depending only on r) such that for every $0 < x < 1$,

$$(6) \qquad 0 < T_{n,2r}(x) < A_r n^r.$$

Calling $p = 2r/\beta$ for a given $\beta > 0$, we have also

$$(7) \qquad \sum_{\nu=0}^{n} |\nu - nx|^{\beta} p_{\nu,n}(x) < A_r^{1/p} n^{\beta/2}.$$

If $\delta = n^{-\alpha}$, $0 < \alpha < \frac{1}{2}$, it is known that [5] for every $l > 0$, there is a constant C where $C = C(\alpha, l)$ such that

$$(8) \qquad \sum_{|\nu n^{-1} - x| > \delta} p_{\nu,n}(x) < Cn^{-l}.$$

3. The linear combination. If $f(x)$ is defined on [0, 1], we define the polynomials

$$\mathfrak{L}_n^{[0]} = [\mathfrak{L}_n^f(x)]^{[0]} = B_n^f(x)$$

$$(9) \qquad (2^k - 1)\, \mathfrak{L}_n^{[2k]} = 2^k \mathfrak{L}_{2n}^{[2k-2]} - \mathfrak{L}_n^{[2k-2]}, \qquad k = 1, 2, \ldots .$$

One can rewrite the relation (9) as

$$\mathfrak{L}_n^{[2k]}(x) = \alpha_k B_{2^k n}(x) + \alpha_{k-1} B_{2^{k-1} n}(x) + \alpha_{k-2} B_{2^{k-2} n}(x) + \ldots + \alpha_0 B_n(x) \tag{10}$$

where by induction, explicit values can be found for the constants α_i, $\alpha_i = \alpha_i(k)$. Note that

$$\alpha_k + \alpha_{k-1} + \alpha_{k-2} + \ldots + \alpha_0 = 1. \tag{11}$$

The polynomial (10) is the linear combination of the ordinary Bernstein polynomials we consider in this paper.

For $r = 1, 2, 3, \ldots, n = 1, 2, 3, \ldots,$ we also define

$$\begin{aligned} \mathfrak{S}_{n,r}^{[0]} &= \mathfrak{S}_{n,r}^{[0]}(x) = S_{n,r}(x) \\ (2^k - 1)\,\mathfrak{S}_{n,r}^{[2k]} &= 2^k \mathfrak{S}_{2n,r}^{[2k-2]} - \mathfrak{S}_{n,r}^{[2k-2]}, \qquad k = 1, 2, \ldots. \end{aligned} \tag{12}$$

Corresponding to the relation (3), for the linear combination (10) we have the following result:

LEMMA 1. *If $f^{(2k+2s)}(x)$ exists at the point x, then*

$$\mathfrak{L}_n^{[2k]}(x) = f(x) + \sum_{r=1}^{2(k+s)} \frac{f^{(r)}(x)}{r!}\,\mathfrak{S}_{n,r}^{[2k]}(x) + \frac{\epsilon_n}{n^{k+s}} \tag{13}$$

where $\epsilon_n \to 0$ as $n \to \infty$.

Proof. We prove this lemma by induction. So we suppose (13) holds and if $f^{(2k+2s+2)}(x)$ exists we show that (13) holds with $2k$ replaced by $2(k+1)$. We have

$$\mathfrak{L}_n^{[2k]}(x) = f(x) + \sum_{r=1}^{2k+2s+2} \frac{f^{(r)}(x)}{r!}\,\mathfrak{S}_{n,r}^{[2k]}(x) + \frac{\epsilon_n}{n^{k+s+1}}.$$

By the relations (9) and (12) we deduce that

$$\begin{aligned} (2^{k+1} - 1)[\mathfrak{L}_n^{[2k+2]}(x) - f(x)] &= 2^{k+1}[\mathfrak{L}_{2n}^{[2k]} - f] - [\mathfrak{L}_n^{[2k]} - f] \\ &= 2^{k+1} \sum_{r=1}^{2k+2s+2} \frac{f^{(r)}(x)}{r!}\,\mathfrak{S}_{2n,r}^{[2k]}(x) - \sum_{r=1}^{2k+2s+2} \frac{f^{(r)}(x)}{r!}\,\mathfrak{S}_{n,r}^{[2k]}(x) + \frac{\epsilon_n}{n^{k+s+1}} \\ &= (2^{k+1} - 1) \sum_{r=1}^{2k+2s+2} \frac{f^{(r)}(x)}{r!}\,\mathfrak{S}_{n,r}^{[2k+2]}(x) + \frac{\epsilon_n}{n^{k+s+1}}. \end{aligned}$$

This establishes the lemma.

We shall now prove the approximation theorem for our linear combination.

THEOREM 1. *If $f(x)$ is defined on $[0, 1]$ with $|f(x)| < M$ and if $f^{(2k)}(x)$ exists at the point x, then*

$$|\,\mathfrak{L}_n^{[2k-2]}(x) - f(x)\,| = O(n^{-k}), \tag{14}$$

and moreover,

$$|\,\mathfrak{L}_n^{[2k]}(x) - f(x)\,| = o(n^{-k}) \text{ as } n \to \infty, \qquad k = 1, 2, \ldots. \tag{15}$$

Proof. By Lemma 1 we have

$$\mathfrak{L}_n^{[2k]} - f = \sum_{r=1}^{2k} \frac{f^{(r)}(x)}{r!} \mathfrak{S}_{n,r}^{[2k]}(x) + \frac{\epsilon_n}{n^k}, \qquad \epsilon_n \to 0 \text{ as } n \to \infty,$$

and if we can show that

$$\sum_{r=1}^{2k} \frac{f^{(r)}(x)}{r!} \mathfrak{S}_{n,r}^{[2k]}(x) = O(n^{-k-1}) \tag{16}$$

then the relation (15) will hold. For this purpose we need

LEMMA 2. *With $\mathfrak{S}_{n,r}^{[2k]}$ defined by* (12),

$$\mathfrak{S}_{n,r}^{[2k]}(x) = 0 \qquad \textit{for } 1 < r < k+1, \tag{17}$$

$$\mathfrak{S}_{n,r}^{[2k]}(x) = O(n^{-k-1}) \qquad \textit{for } r = 1, 2, 3, \ldots. \tag{18}$$

To prove this lemma, we note that by (5)

$$\mathfrak{S}_{n,r}^{[0]}(x) = \phi_{r,r'}(x)\, n^{-(r-r')} + \phi_{r,r'-1}(x)\, n^{-(r-r'+1)} + \ldots + \phi_{r,1}(x)\, n^{-(r-1)}. \tag{19}$$

Applying to n^{-s} the difference operator which connects $\mathfrak{S}^{[2k]}$ with $\mathfrak{S}^{[2k-2]}$ in (12), we obtain

$$2^k(2n)^{-s} - n^{-s} = (2^{k-s} - 1)\, n^{-s}.$$

This is of form an^{-s}, and $a = 0$ if $k = s$. Operating on the right-hand side of (19) with difference operators for $s = 1, 2, 3, \ldots, k$ and omitting vanishing terms we therefore have

$$\mathfrak{S}_{n,r}^{[2k]}(x) = \phi_{k+1}(x)\, n^{-(k+1)} + \ldots + \phi_{r-1}(x)\, n^{-(r-1)}$$

where the $\phi_i(x)$ are polynomials in x independent of n. This proves (18). If $k+1 > r-1$, all terms vanish, and we obtain (17). The lemma is complete.

The relation (15) now follows. By Lemma 1, we find

$$\mathfrak{L}_n^{[2k-2]}(x) - f(x) + \sum_{r=1}^{2k} \frac{f^{(r)}(x)}{r!} \mathfrak{S}_{n,r}^{[2k-2]}(x) + \frac{\epsilon_n}{n^k}$$

and on account of (18), we deduce (14). This establishes the theorem.

In the particular case of the previous theorem for $k = 3$, i.e., if $f^{(6)}(x)$ exists at x, we have by the relations (4) and (13)

$$\lim_{n\to\infty} n^3[\mathfrak{L}_n^{[4]}(x) - f(x)] = \lim_{n\to\infty} n^3\left[\frac{8}{3} B_{4n}^f(x) - 2B_{2n}^f(x) + \frac{1}{3} B_n^f(x) - f(x)\right]$$

$$= \frac{1}{8}(X - 6X^2)\frac{f^{(4)}(x)}{4!} + \frac{5}{4}(1-2x)\, X^2 \frac{f^{(5)}(x)}{5!} + \frac{15}{8} X^3 \frac{f^{(6)}(x)}{6!},$$

and also

$$\lim_{n\to\infty} n^3\, [\mathfrak{L}_n^{[6]}(x) - f(x)]$$

$$= \lim_{n\to\infty} n^3 \left[\frac{64}{21} B_{8n}(x) - \frac{56}{21} B_{4n}(x) + \frac{14}{21} B_{2n}(x) - \frac{1}{21} B_n(x) - f(x)\right] = 0.$$

4. Further theorems on the order of approximation. If $f(x)$ is defined and continuous on $[0, 1]$, then

$$\omega(\delta) = \omega^f(\delta) = \max_{|h|<\delta} |f(x+h) - f(x)|, \quad 0 < x < 1, \quad 0 < x + h < 1$$

is called the modulus of continuity of the function $f(x)$.

THEOREM 2. *If $f^{(2k)}(x)$ exists and is continuous on $[0, 1]$ having a modulus of continuity $\omega_{2k}(\delta)$, then*

$$| \mathfrak{L}_n^{[2k]}(x) - f(x) | < \max \left\{ \frac{C}{n^k} \omega_{2k}(n^{-\frac{1}{2}}), \frac{C'}{n^{k+1}} \right\}$$

where $C = C(k)$ and $C' = C'(k; f)$.

Proof. Since $f^{(2k)}(x)$ exists and is continuous, for x_1, x_2 there is an η, $x_1 < \eta < x_2$ such that

$$f(x_2) - f(x_1) = \sum_{i=1}^{2k} (x_2 - x_1)^i \frac{f^{(i)}(x_1)}{i!} + \frac{(x_2 - x_1)^{2k}}{(2k)!} [f^{(2k)}(\eta) - f^{(2k)}(x_1)].$$

By (11) we have

$$\begin{aligned}
\mathfrak{L}_n^{[2k]} - f &= \sum_{j=0}^{k} \left\{ \alpha_j [B_{2^j n}(x) - f(x)] \right\} \\
&= \sum_{j=0}^{k} \left\{ \alpha_j \sum_{\nu=0}^{2^j n} [f(2^{-j}\nu n^{-1}) - f(x)]\, p_{\nu, 2^j n}(x) \right\} \\
&= \sum_{j=0}^{k} \left\{ \alpha_j \sum_{\nu=0}^{2^j n} \left[\sum_{i=1}^{2k} (2^{-j}\nu n^{-1} - x)^i \frac{f^{(i)}(x)}{i!} \right] p_{\nu, 2^j n}(x) \right\} \\
&\qquad + \sum_{j=0}^{k} \left\{ \alpha_j \sum_{\nu=0}^{2^j n} \left[\frac{(2^{-j}\nu n^{-1} - x)^{2k}}{(2k)!} (f^{(2k)}(\xi_j) - f^{(2k)}(x)) \right] p_{\nu, 2^j n}(x) \right\} \\
&= \Sigma_1 + \Sigma_2,
\end{aligned}$$

where $\xi_j = \xi_j(\nu)$, $x < \xi_j < 2^{-j}\nu/n$, $0 < j < k$. Now

$$\begin{aligned}
\sum_{\nu=0}^{2^j n} \sum_{i=1}^{2k} (2^{-j}\nu n^{-1} - x)^i \frac{f^{(i)}(x)}{i!} p_{\nu, 2^j n}(x) &= \sum_{i=1}^{2k} \sum_{\nu=0}^{2^j n} (2^{-j}\nu n^{-1} - x)^i p_{\nu, 2^j n}(x) \frac{f^{(i)}(x)}{i!} \\
&= \sum_{i=1}^{2k} \mathfrak{S}_{2^j n, i}^{[0]}(x) \frac{f^{(i)}(x)}{i!}.
\end{aligned}$$

Therefore

$$\Sigma_1 = \sum_{i=1}^{2k} \sum_{j=0}^{k} \alpha_j \mathfrak{S}_{2^j n, i}^{[0]}(x) \frac{f^{(i)}(x)}{i!} = \sum_{i=1}^{2k} \mathfrak{S}_{n, i}^{[2k]}(x) \frac{f^{(i)}(x)}{i!},$$

and by Lemma 2 we obtain

$$\Sigma_1 < C' n^{-k-1}.$$

We now evaluate Σ_2. Since for a modulus of continuity, $\omega(\lambda\delta) < (1 + \lambda)\omega(\delta)$ for any $\lambda > 0$ we have

$$\sum_{\nu=0}^{2^j n} \frac{(2^{-j}\nu n^{-1} - x)^{2k}}{(2k)!} \,|\, f^{(2k)}(\xi_j) - f^{(2k)}(x) \,|\, p_{\nu,2^j n}(x)$$

$$< \frac{\omega_{2k}(\delta)}{(2k)!}\left\{ \sum_{\nu=0}^{2^j n} (2^{-j}\nu n^{-1} - x)^{2k} p_{\nu,2^j n}(x) + \frac{1}{\delta}\sum_{\nu=0}^{2^j n} (2^{-j}\nu n^{-1} - x)^{2k} \,|\, \xi_j - x \,|\, p_{\nu,2^j n}(x) \right\}$$

$$< \frac{\omega_{2k}(\delta)}{(2k)!}\left\{ \sum_{\nu=0}^{2^j n} (2^{-j}\nu n^{-1} - x)^{2k} p_{\nu,2^j n}(x) + \frac{1}{\delta}\sum_{\nu=0}^{2^j n} |\, 2^{-j}\nu n^{-1} - x|^{2k+1} p_{\nu,2^j n}(x) \right\},$$

and by (6) and (7), this expression does not exceed

$$\frac{\omega_{2k}(\delta)}{(2k)!}\left\{ \frac{A_k}{(2^j n)^k} + \frac{A'_k}{\delta(2^j n)^{k+\frac{1}{2}}} \right\}.$$

Hence

$$|\, \Sigma_2 \,| < \frac{\omega_{2k}(\delta)}{(2k)!} \sum_{j=0}^{k} |\, \alpha_j \,| \left\{ \frac{A_k}{(2^j n)^k} + \frac{A'_k}{\delta(2^j n)^{k+\frac{1}{2}}} \right\}$$

and putting in particular $\delta = n^{-\frac{1}{2}}$ we have

$$|\, \Sigma_2 \,| < \frac{C}{n^k}\, \omega_{2k}\,(n^{-\frac{1}{2}}).$$

This proves Theorem 2.

COROLLARY. *If $f^{(2k)}(x)$ exists and belongs to* Lip α, $0 < \alpha < 1$, *that is if*

$$|\, f^{(2k)}(x + h) - f^{(2k)}(x) \,| < K \,|\, h \,|^{\alpha},$$

then

$$|\, \mathfrak{L}_n^{[2k]}(x) - f(x) \,| < M n^{-k-\frac{1}{2}\alpha}$$

where M is a constant.

In connection with Theorem 2, we remark that if $f(x)$ is continuous on [0, 1] having a modulus of continuity, $\omega(\delta)$, then for the ordinary $B_n(x)$, Popoviciu **[6]** has shown that

$$|\, B_n^f(x) - f(x) \,| < \tfrac{3}{2}\, \omega(n^{-\frac{1}{2}}).$$

Regarding the case $k = 1$ of the preceding corollary, compare **[2]**.

5. Another property of the $\mathfrak{L}_n^{[2k]}(x)$. If $f(x)$ satisfies only a Lipschitz condition of order α, $0 < \alpha < 1$, then

$$|\, \mathfrak{L}_n^{[2k]}(x) - f(x) \,| < \sum_{j=0}^{k} |\, \alpha_j \,|\, |\, B_{2^j n}(x) - f(x) \,| = O(n^{-\frac{1}{2}\alpha}),$$

and we shall show that this order of approximation cannot in general be improved, that is, one can find functions $f(x) \in$ Lip α $(0 < \alpha < 1)$ such that

$$|\, \mathfrak{L}_n^{[2k]}(x) - f(x) \,| > C_1 n^{-\frac{1}{2}\alpha}$$

with some constant $C_1 > 0$. This shows that in general, $\mathfrak{L}_n^{[2k]}(x)$, $k > 1$ does not approximate $f(x)$ more closely than $B_n^f(x)$. We shall show this for the particular case $k = 1$, the general case $k > 1$ can be treated along similar lines. We have the following theorem.

THEOREM 3. *For every* $0 < \alpha < 1$, *there exist functions* $f(x) \in \mathrm{Lip}\,\alpha$ *such that the order of approximation given by*

$$| \mathfrak{L}_n^{[2]}(x) - f(x) | = O(n^{-\frac{1}{2}\alpha})$$

cannot be improved.

Proof. We shall consider the function $f(x) = |x - x_0|^\alpha$ with fixed $0 < x_0 < 1$ (and where $0 < \alpha < 1$). This function satisfies the Lipschitz condition of order α, namely

$$| x + h - x_0 |^\alpha - | x - x_0 |^\alpha < | h |^\alpha.$$

Now for fixed $0 < x_0 < 1$ and $\gamma > \frac{1}{3}$, for all ν which satisfy

$$(20) \qquad | \nu\mu^{-1} - x_0 | < \mu^{-\gamma}, \qquad 0 < \nu < \mu$$

it is known that [3, p. 133]

$$(21) \qquad R_{\nu,\mu}(x_0) \equiv | \nu\mu^{-1} - x_0 |^\alpha p_{\nu,\mu}(x_0)$$

$$\cong \frac{| \nu\mu^{-1} - x_0 |^\alpha}{[2\pi x_0(1 - x_0)\mu]^{\frac{1}{2}}} \exp\left[- \frac{\mu}{2x_0(1 - x_0)} (\nu\mu^{-1} - x_0)^2 \right] \equiv P_{\nu,\mu}(x_0) \ ;$$

this is a uniform asymptotic relation, that is, uniformly for all ν satisfying (20),

$$\lim_{\mu\to\infty} \frac{R_{\nu,\mu}(x_0)}{P_{\nu,\mu}(x_0)} = 1.$$

We now obtain by a well-known argument [5]

$$R_{\nu,\mu}(x_0) \cong S_{\nu,\mu}(x_0)$$

$$\equiv \mu^{\frac{1}{2}}[2\pi x_0(1 - x_0)]^{-\frac{1}{2}} \int_{\nu\mu^{-1}}^{(\nu+1)\mu^{-1}} | \nu\mu^{-1} - x_0 |^\alpha \exp\left[- \frac{\mu}{2x_0(1 - x_0)} (u - x_0)^2 \right] du$$

uniformly as $\mu \to \infty$ for all ν satisfying (20). Now

$$| \nu\mu^{-1} - x_0 |^\alpha - | u - x_0 |^\alpha = O\left(| \nu\mu^{-1} - u |^\alpha \right) = O\left(\mu^{-\alpha}\right)$$

uniformly in ν and u as $|\nu\mu^{-1} - u| < \mu^{-1}$, and so we deduce

$$(22) \ R_{\nu,\mu}(x_0) \cong S_{\nu,\mu}(x_0)$$

$$= \mu^{\frac{1}{2}}[2\pi x_0(1 - x_0)]^{-\frac{1}{2}} \int_{\nu\mu^{-1}}^{(\nu+1)\mu^{-1}} | u - x_0 |^\alpha \exp\left[- \frac{\mu}{2x_0(1 - x_0)} (u - x_0)^2 \right] du$$

$$+ O\left[\mu^{-\alpha} \mu^{\frac{1}{2}}[2\pi x_0(1 - x_0)]^{-\frac{1}{2}} \int_{\nu\mu^{-1}}^{(\nu+1)\mu^{-1}} \exp\left[- \frac{\mu}{2x_0(1 - x_0)} (u - x_0)^2 \right] du \right]$$

uniformly in ν satisfying (20), as $\mu \to \infty$.

Since $|\nu\mu^{-1} - x_0|^\alpha < 1$ and applying (8) we have

$$\sum_{|\nu\mu^{-1}-x_0|>\delta_0} |\nu\mu^{-1} - x_0|^\alpha p_{\nu,\mu}(x_0) < \sum_{|\nu\mu^{-1}-x_0|>\delta_0} p_{\nu,\mu}(x_0) < C_2\mu^{-l},$$

where $\delta_0 = \mu^{-\gamma}$ for every $l > 0$ with $0 < \gamma < \frac{1}{2}$, and where the constant $C_2 = C_2(\gamma, l)$. We take $l > \frac{1}{2}\alpha$. Then for $\frac{1}{3} < \gamma < \frac{1}{2}$,

$$\begin{aligned}\sum_{\nu=0}^{\mu} R_{\nu,\mu}(x_0) &= \sum_{|\nu\mu^{-1}-x_0|<\delta_0} R_{\nu,\mu}(x_0) + \sum_{|\nu\mu^{-1}-x_0|>\delta_0} R_{\nu,\mu}(x_0) \\ &= \sum_{|\nu\mu^{-1}-x_0|<\delta_0} R_{\nu,\mu}(x_0) + O(\mu^{-l}) \\ &= (1 + \epsilon_\mu) \sum_{|\nu\mu^{-1}-x_0|<\delta_0} S_{\nu,\mu}(x_0) + o(\mu^{-\frac{1}{2}\alpha})\end{aligned}$$

where $\epsilon_\mu \to 0$ for $\mu \to \infty$. We now obtain

$$\begin{aligned}|f(x_0) - \mathfrak{L}_n^{[2]}(x_0)| &= |\mathfrak{L}_n^{[2]}(x_0)| = |2B_{2n}(x_0) - B_n(x_0)| \geqslant 2B_{2n}(x_0) - B_n(x_0) \\ &= 2\sum_{\nu=0}^{2n} R_{\nu,2n}(x_0) - \sum_{\nu=0}^{n} R_{\nu,n}(x_0) \\ &= (2 + \epsilon'_n) \sum_{|\nu(2n)^{-1}-x_0|<\delta_2} S_{\nu,2n}(x_0) - (1 + \epsilon''_n) \sum_{|\nu n^{-1}-x_0|<\delta_1} S_{\nu,n}(x_0) + o(n^{-\frac{1}{2}\alpha}),\end{aligned}$$

where $\delta_2 = (2n)^{-\gamma}$, $\delta_1 = n^{-\gamma}$, and $\epsilon'_n \to 0$ and $\epsilon''_n \to 0$ as $n \to \infty$. Applying (22), this expression is seen to be equal to

$$\begin{aligned}&\frac{2 + \epsilon'_n}{\sqrt{\pi}} \frac{[x_0(1 - x_0)]^{\frac{1}{2}\alpha}}{n^{\frac{1}{2}\alpha}} \int_0^{2^{-\gamma}n^{\frac{1}{2}-\gamma}[x_0(1 - x_0)]^{-\frac{1}{2}}} v^\alpha \exp(-v^2)\, dv \\ &\quad + O\left[\frac{1}{(2n)^\alpha} \int_0^{2^{-\gamma}n^{\frac{1}{2}-\gamma}[x_0(1 - x_0)]^{-\frac{1}{2}}} \exp(-v^2)\, dv\right] \\ &\quad - \frac{1 + \epsilon''_n}{\sqrt{\pi}} \frac{[2x_0(1 - x_0)]^{\frac{1}{2}\alpha}}{n^{\frac{1}{2}\alpha}} \int_0^{n^{\frac{1}{2}-\gamma}[2x_0(1 - x_0)]^{-\frac{1}{2}}} v^\alpha \exp(-v^2)\, dv \\ &\quad + O\left[n^{-\alpha} \int_0^{n^{\frac{1}{2}-\gamma}[2x_0(1 - x_0)]^{-\frac{1}{2}}} \exp(-v^2)\, dv\right] + o(n^{-\frac{1}{2}\alpha}),\end{aligned}$$

where $\frac{1}{3} < \gamma < \frac{1}{2}$. But the second and fourth terms need not be considered, as they are of order $O(n^{-\alpha})$; the integrals in the remaining two terms converge to the same positive limit, and the difference of the factors outside these integrals is of the form

$$C_3 n^{-\frac{1}{2}\alpha} + o(n^{-\frac{1}{2}\alpha})$$

where C_3 is positive as $0 < \alpha \leqslant 1$. So we deduce

$$|f(x_0) - \mathfrak{L}_n^{[2]}(x_0)| \geqslant C_4 n^{-\frac{1}{2}\alpha}$$

where C_4 is a strictly positive constant, proving the theorem.

We have constructed linear combinations of the Bernstein polynomials $B_n(x)$, namely

$$\mathfrak{L}_n^{[2k]}(x),$$

of degree $2^k n$, which under conditions imposed on the corresponding function, approach $f(x)$ more closely than

$$B_{2^k n}(x).$$

The order of approximation of a function by polynomials of best approximation is generally better than that given by the

$$\mathfrak{L}_n^{[2k]}(x).$$

For instance, if $f^{(2k)}(x) \in \text{Lip}\ \alpha$, $0 < \alpha \leqslant 1$, there are polynomials $P_n(x)$ of degree n such that

$$| P_n(x) - f(x) | \leqslant M' n^{-2k-\alpha}$$

[4, p. 18]. For the

$$\mathfrak{L}_n^{[2k]}(x)$$

of degree $2^k n$ we have

$$| \mathfrak{L}_n^{[2k]}(x) - f(x) | \leqslant M n^{-k-\frac{1}{2}\alpha}.$$

It remains an open question whether there are other linear combinations of degree not exceeding $2^k n$ approaching $f(x)$ more closely than the combination

$$\mathfrak{L}_n^{[2k]}(x).$$

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REPRESENTATION OF LINEAR FUNCTIONALS ON KÖTHE SPACES

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1. Introduction. Köthe spaces, in the terminology of Diendonné [**2**], are certain spaces X of real valued integrable functions. In this paper we consider the problem of representation of continuous linear functionals on vector valued Köthe spaces. The elements of a Köthe space $X(B)$ are functions $\mathbf{f}(t)$, $0 < t < 1$, with values in a Banach space B (see §2). In the case $X = L^p$, this problem was solved by Dieudonné [**3**]. In May 1952, the second of the present authors found that Dieudonné's methods also apply to spaces[1] $\Lambda(\phi, p)$ for $p > 1$. However, difficulties arise even for spaces $\Lambda(\phi, 1)$, since Dieudonne's methods depend heavily on the reflexivity of the space X. This motivates an entirely new approach to the problem proposed here which is applicable for more general spaces X. Our main idea is the use of linear operators of class (b, o) of Kantorovitch [**6**; **8**], which seem to provide the most natural way of handling the problem. In fact, this method is applicable also to cross-spaces $B \otimes_\delta X$ (see Schatten [**12**]) with a certain cross-norm δ, not symmetric with respect to B and X. Köthe spaces $X(B)$ are special cases of these $B \otimes_\delta X$. In order not to complicate the exposition, we confine our attention to the simpler case of Köthe spaces; functionals on cross-spaces will be discussed in a separate paper.

In §2 we give the definition of a Köthe space X and consider its properties as an abstract Banach lattice. Care must be taken not to exclude spaces such as $X = L^1$ for which the conjugate space X^* does not satisfy condition (f) of §2. Therefore these properties (which must also hold for X^*) turn out to be partly weaker than those given in [**8**, p. 215]. In §3 we give our main results concerning linear functionals on spaces $X(B)$.

2. Köthe spaces and Banach lattices. Let C be a non-empty class of positive integrable functions $c(t)$. The Köthe space $X = X_C$ consists of all measurable functions $f(t)$ for which

$$(1) \qquad \|f\|_X = \sup_{c \in C} \int_0^1 |f(t)|\, c(t)\, dt < +\infty.$$

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[1]Spaces $\Lambda(\phi, p)$ for $\phi = x^{a-1}$ were defined by Lorentz [**10**]; in his talks in Tübingen (1948) and Kingston (1950) he indicated the generalization to an arbitrary decreasing ϕ [**11**]. Statements made in this connection in [**13**, p. 273] are misleading: the manuscipt mentioned there was written after these talks; in the 1950 Report of the Summer Research Institute, Halperin gives its content as obtained jointly with Lorentz.

Without changing X_C or the value of $\|f\|$ we may assume that

(a) C is ***normal***: if $c \in C$ and $0 \leq c_1(t) \leq c(t)$, then $c_1 \in C$;

(b) C is ***convex***: if $c_i \in C$, $0 \leq \alpha_i \leq 1$, $\Sigma_1^n \alpha_i = 1$, then $\Sigma \alpha_i c_i \in C$.

We shall also assume that

(c) if $c_n \in C$ and $c_n(t) \uparrow c(t)$, then $c \in C$;

(d) $\mathbf{1} \in C$, where $\mathbf{1}(t) = 1$ a.e. on $(0, 1)$;

(e) $$\int_0^1 c(t)\, dt \leq 1, c \in C.$$

Conditions (a), (d), (e) imply that all measurable functions $c(t)$ with $0 \leq c(t) \leq 1$ belong to C, condition (e) that all bounded functions belong to X_C. From the definition of $\|f\|$ it follows that X_C is normal: $f \in X$ and $|g(t)| \leq |f(t)|$ imply $g \in X$. Condition (d) and (1) also imply that all functions $f \in X_C$ are integrable and that

(2) $$\int_0^1 |f(t)|\, dt \leq \|f\|_X.$$

It is easy to see that under these assumptions X_C is a Banach space. General Köthe spaces of integrable functions were considered by Dieudonné [**2**]. Köthe and Köthe and Toeplitz in a series of papers beginning in 1934 were dealing with spaces of sequences of similar type. Dieudonné defines the Köthe space X_C to consist of all $f \in L^1$ with $\int |f|\, c\, dt < +\infty$ for each $c \in C$; under the assumptions (a), (b), and (c) this is equivalent to our definition (given also in [**11**]). In fact, if $\int |f|\, c\, dt$, $c \in C$ is not bounded for some f, there is a $c \in C$ with $\int |f|\, c\, dt = +\infty$. For the proof, we choose $c_n \in C$ with $\int |f|\, c_n\, dt > n^3$ $(n = 1, 2, \ldots)$; then $c = \Sigma n^{-2} c_n$ belongs to C by (a), (b), and (c), and $\int |f|\, c\, dt > n$ for each n. On the other hand, if the conditions (a), (b), (c) are not assumed, Dieudonné's definition is more general than that given above. In what follows a Köthe space X_C is always a space with C satisfying (a)–(e).

In X we introduce a partial ordering, writing $f \leq g$ if $f(t) \leq g(t)$ a.e. With this order X becomes a Banach lattice [**1**; **5**; **6**; **8**] with the following properties:

(I) $f \geq 0$, $g \geq 0$ imply $f + g \geq 0$;

(II) $f \geq 0$, $a \geq 0$ imply $af \geq 0$;

(III) each finite set $E \subset X$ is bounded from above;

(IV) each set $E \subset X$ bounded from above has a supremum $f_0 = \bigcup f = \sup f$ in X.

Property (IV) follows from the fact that the space S of all measurable functions has this property. The order induces the *order-convergence* (*o-convergence*) $f_n \to f$ (o), which for the space X_C is equivalent to $f_n(t) \to f(t)$ a.e. and to the existence of a $g \in X$ with $|f_n| \leq g$.

From (1) we deduce also:

(V) $|f| \leq |g|$ implies $\|f\| \leq \|g\|$;

(VI) if $0 \leq f_n \in X$, $f_n \uparrow f$ (where $f \in X$ or f is the element $+\infty$), then $\|f_n\| \to \|f\|$. (In particular, $f_n \uparrow +\infty$ implies $\|f_n\| \to \infty$.)

A space X_C has a weak form of regularity:

(VII) each set $E \subset X$ has a denumerable subset $E' \subset E$ with $\sup E' = \sup E$. (This also follows from the corresponding property of S.)

We finally note that a Köthe space X_C has a (weak) unit:

(VIII) there is an element $\mathbf{1} \in X$ such that $f \cap n\mathbf{1} \uparrow f$ for each $0 < f \in X$.

The following lemma is implicitly contained in [**8**, pp. 201–203]:

LEMMA. *For a Banach lattice X satisfying* I–VIII, *the o-boundedness of a set $E \subset X$ and the boundedness of the set*

$$H = \left\{ \left\| \bigcup_{i=1}^{k} |f_i| \right\| \right\},$$

$f_i \in E$ are equivalent. Moreover, with $g = \sup |f|$, where the supremum is taken for all $f \in E$, and $A = \sup H$ we have $A = \|g\|$.

First let $A < +\infty$. Each of the sets $E_n = \{|f| \cap n\mathbf{1}\}$ is bounded and for the $g_n = \sup E_n$ we have $g_n \uparrow g = \sup |f|$ (with $g \in X$ or $g = +\infty$). By VII,

$$g_n = \bigcup_{k=1}^{\infty} (|f_{kn}| \cap n\mathbf{1})$$

for properly chosen $f_{kn} \in E$. By VI,

$$\| g_n \| = \lim_{k\to\infty} \left\| \bigcup_{i=1}^{k} |f_{in}| \cap n\mathbf{1} \right\| \leq \lim \left\| \bigcup_{i=1}^{k} |f_{in}| \right\| \leq A.$$

By VI, this shows that $g \in X$ and that E is bounded. We also have $\|g\| \leq A$. The converse is obvious.

The *dual space* X' of a Köthe space X_C is the set of all measurable g such that

$$\| g \|_{X'} = \sup_{\|f\| \leq 1} \int_0^1 |g| \, |f| \, dt < +\infty.$$

In other words, $X' = X_{C'}$, where C' consists of all elements $f \in X$ with $0 \leq f$, $\|f\| \leq 1$. It follows from (a)–(e) and VI that C' also satisfies (a)–(e). Clearly, X' is a subspace of the conjugate space X^*. A Köthe space X is *perfect* if $(X')' = X$; a reflexive space X is perfect, but not conversely (example: $X = L^1$).

THEOREM 1. *If X is a separable perfect Köthe space, then X' is identical with X^*.*

Proof. In virtue of Lorentz [**11**, Theorem 3], X' is identical with X^* if $X = X_C$ satisfies the condition

(f) if $f \in X$ and χ_e is the characteristic function of the set e, then $\|f\chi_e\| \to 0$ with $me \to 0$.

Hence we have to show that a separable perfect space X_C has the property (f). The following proof uses an argument due to Köthe [**9**, pp. 105–106].

We have $Y' = X$, and X is separable. It follows (by the usual method of a diagonal subsequence) that the unit sphere U in Y is sequentially weakly compact (in the weak topology generated by X). If (f) is not fulfilled for X, there

is an $f \in X$ and a sequence of sets e_n with $me_n \to 0$ such that $\|f\chi_{e_n}\| > 1$. Then for each n there is a function $c_n \in U$ such that $\|c_n\|_Y < 1$ and

$$\int_0^1 |f| \chi_{e_n} c_n \, dt > \tfrac{1}{2}.$$

The functions $\bar{c}_n = c_n \chi_{e_n}$ have the properties $\|\bar{c}_n\|_Y < 1$,

$$\int_0^1 |f| \bar{c}_n \, dt > \tfrac{1}{2}$$

and $\bar{c}_n(t) = 0$ outside of the set e_n. By passing to a subsequence of the $\bar{c}_n$, if necessary, we may assume that the $\bar{c}_n$ are weakly convergent to $\bar{c}$, say, and that $\Sigma me_n < +\infty$. Then the set lim sup e_n is of measure zero. Let e be disjoint with $\bigcup_{n>N} e_n$; then

$$\int_e \bar{c}(t) \, dt = \int_0^1 \chi_e \bar{c} \, dt = \lim_{n\to\infty} \int_0^1 \chi_e c_n \, dt = 0,$$

so that $\bar{c}$ vanishes outside of $\bigcup_{n>N} e_n$. It follows that $\bar{c}$ vanishes a.e. But this is a contradiction:

$$0 = \int_0^1 |f| \bar{c} \, dt = \lim_{n\to\infty} \int_0^1 |f| \bar{c}_n \, dt > \tfrac{1}{2}.$$

This completes the proof of Theorem 1.

It is easy to check that the spaces $\Lambda(\phi, p)$ satisfy (a)–(f) (compare [**10**; **11**]). The spaces $\Lambda(\phi, 1)$ are perfect [**10**], the spaces $\Lambda(\phi, p)$, $p > 1$ are reflexive [**11**].

Let B be a Banach space and $X = X_C$ a Köthe space of real-valued functions. The space $X(B)$ by definition consists of all functions $\mathbf{f}(t)$ from $(0, 1)$ to B which are weakly measurable and such that $\|\mathbf{f}(t)\|$ belongs to X. For separable spaces X, $\|\mathbf{f}(t)\|$ is measurable whenever $\mathbf{f}(t)$ is weakly measurable [**3**]. We put

$$\|\mathbf{f}\|_{X(B)} = \| \, \|\mathbf{f}(t)\|_B \, \|_X = \sup_{c \in C} \int_0^1 \|\mathbf{f}(t)\| \, c(t) \, dt. \tag{3}$$

It is easy to show that $X(B)$ is a Banach space. If B is separable and X_C satisfies condition (f), $X(B)$ is also separable, since elements of the type

$$\sum_{i=1}^{n} x_i \chi_{e_i}(t)$$

with $x_i \in B$ and measurable and disjoint sets e_i are everywhere dense in $X(B)$.

3. Linear operators of class (b, o) and continuous linear functionals on $X(B)$. Let B be a Banach space and X a Banach lattice. A linear mapping $f = U(x)$ from B to X is called an operator of class (b, o) if the set of the $|U(x)|$ for all $\|x\| < 1$ is o-bounded in X. Then

$$|U| = \bigcup_{\|x\|<1} |U(x)| \tag{4}$$

is the "abstract norm" of U. (In [**8**], U is defined to be of class (b, o) if $x_n \to x$

always implies $U(x_n) \to U(x)$ (o). From the lemma of §2 and [**8**, pp. 202, 187] it follows that a Banach lattice X satisfying I–VIII is a K^+ space, and then both definitions are equivalent [**8**, p 258]). We shall use sometimes the notation $\langle x, g\rangle$ for the value of the functional $g \in B^*$ at $x \in B$.

The Lemma of §2 shows that in case when X satisfies I–VIII, a linear operator U is of class (b, o) if and only if

$$(5) \qquad A = \sup_{\|x_i\| \leq 1} \| U(x_1) \cup U(x_2) \cup \ldots \cup U(x_n) \| < +\infty.$$

THEOREM 2. *The general form of an operator $f = U(x)$ of class (b, o) from a separable Banach space to a Köthe space X_C is given by*

$$(6) \qquad f(t) = \langle x, \mathfrak{g}(t)\rangle$$

where $\mathfrak{g}(t)$ belongs to $X_C(B^*)$. Moreover, $|U|$ is the function $\|\mathfrak{g}(t)\|_{B^*}$.

Proof. For $X = L^p$, $p > 1$, this was given by Kantorovitch and Vulich [**7**, Theorem 14; or **8**, p. 330], except for the last statement.

If U is an operator of desired class, then it is also an operator of class (b, o) from B to L^1. By the above theorem with $X = L^1$, $f(t) = \langle x, \mathfrak{g}(t)\rangle$ for each $x \in B$ and almost all t, where $\mathfrak{g}(t)$ is weakly measurable and $\|\mathfrak{g}(t)\| \in L^1$. We have

$$|\langle x, \mathfrak{g}(t)\rangle| = |f(t)| \leq |U|(t)\|x\| \qquad \text{a.e.};$$

this relation holds for each x and all t except for a set E_x of measure zero, which may depend on x. Since B is separable, it is easy to prove that there is a set E of measure zero such that

$$|\langle x, \mathfrak{g}(t)\rangle| \leq |U|(t)\|x\|, \qquad x \in B,\ t \notin E.$$

Then for $t \notin E$, $\|\mathfrak{g}(t)\| \leq |U|(t)$. This shows that $\mathfrak{g}(t) \in X(B^*)$.

On the other hand, from (6) we derive

$$|f(t)| = |\langle x, \mathfrak{g}(t)\rangle| \leq \|\mathfrak{g}(t)\| \ \|x\|,$$

so that the conditions are sufficient and $|U|(t) \leq \|\mathfrak{g}(t)\|$. This completes the proof.

If X is not separable, a representation formula for an operator of class (b, o) can still be given under stronger assumptions on X_C. However, this will not be used for our main theorems, and we do not give full proofs. Here also, the case $X_C = L^p$ has been discussed by Kantorovitch and Vulich [**7**].

We shall formulate the following properties of a class $C = \{c\}$:

(g) C is *average-invariant:* if $c \in C$, $e \subset (0, 1)$ and if $\bar{c}$ is obtained from c by replacing its values on e by the average $(me)^{-1}\int_e c\, dt$, then $\bar{c} \in C$.

(h) There is a function $A(\epsilon) > 0$, $A(\epsilon) \to \infty$ for $\epsilon \to 0$ such that for each set e of measure $me < \epsilon$ there is a $c \in C$ with $c(t) \geq A(\epsilon)$ on e.

This allows us to characterize the integrals

$$F(t) = \int_0^t f(u)\, du$$

of functions $f \in X_C$. We say that a real function $F(t)$ on $(0, 1)$ is of bounded X_C-variation if, for all subdivisions $0 = t_0 < t_1 < \ldots < t_n = 1$ of $(0, 1)$, the functions $\bar{F}(t)$ defined by

$$\bar{F}(t) = \frac{1}{t_{i+1} - t_i} [F(t_{i+1}) - F(t_i)], \qquad t \in (t_i, t_{i+1}) \ (i = 0, 1, \ldots, n-1)$$

have uniformly bounded norms $||\bar{F}||_X$. Let X_C be a Köthe space. Then a function $F(t)$ with $F(0) = 0$ is an integral of a function $f \in X_C$ if F is of bounded X_C-variation and if (h) holds. Conversely, any such integral is of bounded X_C-variation if (g) holds. The proof is similar to the proof of F. Riesz's theorem on integrals of functions $f \in L^p$. By means of this result we obtain

THEOREM 3.[2] *If the Köthe space X_C satisfies* (g) *and* (h), *the general form of an operator $f = U(x)$ of class (b, o) from a Banach space B to X_C is given by*

$$f(t) = \frac{d}{dt} \langle x, \mathbf{G}(t) \rangle$$

where $\mathbf{G}(t)$ is a mapping from $(0, 1)$ *to B^* such that the functions*

$$\bar{G}(t) = \frac{1}{t_{i+1} - t_i} || \mathbf{G}(t_{i+1}) - \mathbf{G}(t_i) ||_{B^*}, \qquad t \in (t_i, t_{i+1}) \ (i = 0, 1, \ldots, n-1)$$

have uniformly bounded norms in X.

The proof consists in a direct construction of the function $|U|(t)$.

We now come to our main subject: to linear continuous functionals on spaces $X_C(B)$. We assume throughout the rest of this section that B is separable, that X satisfies condition (f), and that X' and X^* are identical. Theorem 1 provides examples of such X—all perfect separable Köthe spaces belong to this class. More particular examples of such X are spaces $\Lambda(\phi, p)$, which reduce to L^p for $\phi(x) = 1$.

THEOREM 4. *There is a natural isomorphism between the spaces of* (i) *all continuous linear functionals $L(f)$ on $X(B)$;* (ii) *all functions $\mathfrak{g}(t)$ belonging to $X^*(B^*)$;* (iii) *all (b, o)-operators $U(x)$ mapping B into X^*. The general form of a continuous linear functional $L(\mathbf{f})$ on $X(B)$ is given by*

$$L(\mathbf{f}) = \int_0^1 \langle \mathbf{f}(t), \mathfrak{g}(t) \rangle \, dt, \tag{7}$$

where $\mathfrak{g}(t)$ belongs to $X^*(B^*)$ and $||L|| = ||\mathfrak{g}||$.

Proof. To each $\mathfrak{g}$ the relation (7) lets correspond an $L(\mathbf{f})$ which is clearly a continuous linear functional on $X(B)$ with norm $||L|| < ||\mathfrak{g}||$. Theorem 2 establishes a (1, 1) correspondence between the U and the $\mathfrak{g}$. We shall show that to each L there corresponds a U. For a fixed $x \in B$ and variable $f \in X$, $L(xf)$ is a continuous linear functional on X, which is characterized by a function

[2]Added January 28, 1953.

$g(t)$ from X'. The mapping $x \to g$ defines a linear operator $U(x) = g$. The correspondence between x and g is given by

$$(8) \qquad L(xf) = \int_0^1 f(t)\, g(t)\, dt.$$

To show that U is of class (b, o), we shall prove (5) with $A < \|L\|$. We have to show that for each finite set of elements $x_i \in B$ $(i = 1, \ldots, n)$ with $\|x_i\| < 1$ we have $\|g\| < \|L\|$ where $g = \bigcup_{i=1}^n |g_i|$, $g_i = U(x_i)$. There are disjoint measurable sets e_i with $\bigcup_1^n e_i = (0, 1)$ and $g(t) = |g_i(t)|$ on the set e_i. Put $\epsilon_i(t) = \operatorname{sign} g_i(t)$, so that $\epsilon_i(t) = 0$ outside of e_i. Let $f \in X_C$ be arbitrary and

$$\mathbf{f}(t) = \sum_{i=1}^n x_i f(t)\, \epsilon_i(t).$$

Then by (8)

$$(9) \qquad \int_0^1 fg\, dt = \sum_i \int_{e_i} f\, |g_i|\, dt = \sum_i L(x_i f \epsilon_i) = L(\mathbf{f}).$$

On the other hand,

$$\begin{aligned} |L(\mathbf{f})| &< \|L\| . \|\mathbf{f}\| = \|L\| \Big\| \, \|\sum_i x_i f \epsilon_i\|_B \Big\|_X \\ &= \|L\| . \Big\| \sum_i \|x_i\| \, |f| \, |\epsilon_i| \Big\| \\ &< \|L\| . \Big\| \sum_i \|x_i\| \, |f| \, |\chi_{e_i}| \Big\| = \|L\| . \|\mathbf{f}\|_X . \end{aligned}$$

Comparing this with (8) we obtain $\|g\| < \|L\|$.

Now let

$$(10) \qquad (U(x))(t) = \langle x, \mathfrak{g}(t) \rangle, \qquad \mathfrak{g} \in X(B^*)$$

be the representation of U given by Theorem 2. For elements $\mathbf{f}$ of $X(B)$ of the form

$$\mathbf{f} = \sum_{i=1}^n x_i f_i, \qquad x_i \in B, f_i \in X$$

we have by (8) and (9)

$$(11) \qquad \begin{aligned} L(\mathbf{f}) &= \sum_i L(x_i f_i) = \sum \int_0^1 f_i\, U(x_i)\, dt \\ &= \int_0^1 \sum_i f_i(t) \langle x, \mathfrak{g}(t) \rangle\, dt = \int_0^1 \langle \mathbf{f}(t), \mathfrak{g}(t) \rangle\, dt. \end{aligned}$$

Since both L and the last integral in (11) are continuous functionals on $X(B)$ and the set of the $\mathbf{f}$ of the above kind is everywhere dense in $X(B)$, we obtain (7) for an arbitrary $\mathbf{f} \in X(B)$.

We also have $\| \, |U| \, \| = \|g\| > \|L\| > A = \| \, |U| \, \|$ which proves that $\|L\| = \|g\|$. This completes the proof. Theorem 4 may also be stated in the form $(X(B))^* = X^*(B^*)$. As a corollary we have:

THEOREM 5. *If the spaces B, X are separable and reflexive, then so is $X(B)$.*

In particular, the vector-valued spaces $\Lambda(\phi, p, B)$, $p > 1$ are reflexive. In the case when B is a finite-dimensional Euclidean space, this was also proved by Ellis and Halperin [**4**].

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FUNCTION SPACES DETERMINED BY A LEVELLING LENGTH FUNCTION

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1. Introduction. In this paper we introduce the function spaces L^λ and $L^\lambda(B)$ which generalize the classical L^p and $L^p(B)$ spaces respectively. For those λ which possess what we call the *levelling* property we give a discussion of the conjugate space to $L^\lambda(B)$; our treatment applies, in particular, to the $L_{(w)}{}^p(B)$ and $M_{(w)}{}^q(B)$ spaces defined in [**7**, §2]. (*See also the note at end of this paper.*)

When specialized to apply to the classical $L^p(B)$ case, our methods are closely related to those used previously by other writers, including Bochner and Taylor [**3**], Pettis [**9**; **10**], Phillips [**11**; **12**], and Dieudonné [**4**]; but even for the classical case, our treatment is, we believe, simpler, more direct, and more complete than any given previously. (We are indebted to J. Dieudonné for pointing out that our own results could be strenghtened by applying Eberlein's Theorem and for a reference to the papers by Phillips.)

2. Terminology. Throughout this paper we shall use without comment the terminology given in [**7**, §2] as well as the following:

λ is called a *length function* if for every measurable function u with $0 \leqslant u(P) \leqslant \infty$ for almost all P, $\lambda(u)$ is defined with $0 \leqslant \lambda(u) \leqslant \infty$ and satisfies:

(L 1) $\lambda(u) = 0$ *whenever* $u(P) = 0$ *for almost all* P,

(L 2) $\lambda(u) \leqslant \lambda(u_1)$ *whenever* $u(P) \leqslant u_1(P)$ *for all* P,

(L 3) $\lambda(u_1 + u_2) \leqslant \lambda(u_1) + \lambda(u_2)$,

(L 4) $\lambda(ku) = k\lambda(u)$ *for all* $k > 0$,

(L 5) $u_1(P) \leqslant u_2(P) \leqslant \ldots$ *for all* P *implies* $\lambda(\sup u_n) = \sup \lambda(u_n)$.

We frequently write $\lambda(E)$ in place of $\lambda(1_E)$. S is called *coarse* if for some $A > 0$, $\gamma(e) < A$ implies $\gamma(e) = 0$; if S is not coarse we define $\lambda(+0)$ as inf $\lambda(e)$ for all e with $\gamma(e) > 0$. A set E is called λ-*null* if $\lambda(u) = 0$ whenever u vanishes outside E and E is called λ-*purely-infinite* if $\gamma(E) > 0$ and $\lambda(u) = \infty$ whenever $u(P) > 0$ on some subset of positive measure contained in E.

A length function λ is called *continuous* if

(L 6) $\lambda(u) = \sup_e \lambda(u_e)$.

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A continuous λ is called a *levelling* length function if it satisfies the following two conditions:

(L 7) *$\lambda(e) < \infty$ for some e with $\gamma(e) > 0$, and $\lambda(e) > 0$ for some e,*

(L 8) $$\lambda\left(\sum_{i=1}^{m} k_{e_i i}\right) \geqslant \lambda\left(\left(\frac{k_1\gamma(e_1) + k_2\gamma(e_2)}{\gamma(e_1 + e_2)}\right)_{e_1+e_2} + \sum_{i=3}^{m} k_{e_i i}\right)$$

for all $m \geqslant 2$, disjoint $e_1, \ldots, e_m$ with all $\gamma(e_i) > 0$, and arbitrary $k_i \geqslant 0$.

Other possible conditions on λ are listed for reference as follows:

(L 9) *Either S is coarse or $\lambda(+0) = 0$.*

(L 10) *$\lambda(u) < \infty$ implies $u = u_E$ for some E which is a countable union of sets of finite measure.*

(L 11) *$\lambda(u) = \lambda(u_E) < \infty$ implies $u = u_E$.*

(L 12) *$\lambda(u) < \infty$, $\epsilon > 0$ imply $\lambda(u - u_e) < \epsilon$ for some e.*

(L 13) *$\lambda(u) < \infty$ implies $\lambda(u - u_N) \to 0$ as $N \to \infty$.*

For every length function λ we define the *conjugate length function* λ^* (sometimes denoted by μ) by the relation

$$\lambda^*(v) = \sup \int u(P)v(P)d\gamma(P),$$

for all u with $\lambda(u) \leqslant 1$. Notation such as (L 9)* refers to λ^* in place of λ.

For a given weight function w on S as described in [**7**, §2], $\lambda_{(w)}{}^p$ and $\mu_{(w)}{}^q$ shall denote the length functions:

$$\lambda_{(w)}{}^p(u) = \begin{cases} \left(\int_0^{\gamma} u^*(x)^p w^*(x)\,dx\right)^{1/p} & 1 \leqslant p < \infty, \\ u^*(0) = \text{ess. sup}\, u(P), & p = \infty; \end{cases}$$

$$\mu_{(w)}{}^q(v) = \begin{cases} \left(\int_0^{\gamma} \left(\frac{v^{*\circ}(x)}{w^*(x)}\right)^q w^*(x)\,dx\right)^{1/q} & 1 \leqslant q < \infty, \\ \sup\left(\frac{v^{*\circ}(x)}{w^*(x)}\right), & q = \infty. \end{cases}$$

If $f(P)$ has $|f(P)|$ measurable then $\lambda(f)$ will mean $\lambda(u)$ with $u(P) = |f(P)|$; L^λ will denote the space of numerical valued f which are measurable and have $\lambda(f)$ finite; $L^\lambda(B)$ will denote the space of *Bochner measurable* f, valued in B and with $\lambda(f)$ finite. L^λ, $L^\lambda(B)$ will sometimes be denoted by $L_{(w)}{}^p$, $L_{(w)}{}^p(B)$, respectively, when $\lambda = \lambda_{(w)}{}^p$ and by $M_{(w)}{}^q$, $M_{(w)}{}^q(B)$, respectively, when $\lambda = \mu_{(w)}{}^q$.

F (and similarly G) will always denote an additive set function defined for every e, with $F(e) = 0$ whenever $\gamma(e) = 0$; we define

$$\lambda(F) = \sup \lambda\left(\sum_{i=1}^{m}\left(\frac{|F(e_i)|}{\gamma(e_i)}\right)_{e_i}\right)$$

for all finite collections of disjoint e_i with all $\gamma(e_i) > 0$, and

$$|F|(e) = \sup (|F(e_1)| + \ldots + |F(e_m)|)$$

for all finite collections of disjoint $e_i \subset e$. For given G we let $V = V(G)$ denote the closed linear space determined by the values of the $G(e)$. G is said to be *majorized* on E (by the constant $M < \infty$) if $|G(e)| \leqslant M\,\gamma(e)$ for every $e \subset E$ (see Phillips [**12**, p. 133], Dieudonné [**4**, p. 130]).

We list a further possible condition on λ, namely:

(*L* 10)′ $\lambda(F) < \infty$ *implies* $F = F_E$ *for some* E *which is a countable union of sets of finite measure.*

In §6 we consider numerical valued $\tilde{g} = g(c) = g(c, P)$ which, for fixed c, is defined for almost all P (the exceptional set depending on c) and is measurable and satisfies: $g(c_1, P) + g(c_2, P) = g(c_1 + c_2, P)$ for almost all P, for all c_1, c_2; $g(kc, P) = kg(c, P)$ for almost all P, for every scalar k and every c. We define

$$\lambda(\tilde{g}) = \sup \lambda\left(\sum_{i=1}^{m} g(c_{e,i})\right)$$

for all finite collections of disjoint e_i and arbitrary c_i with $|c_i| = 1$. (We use the convention $g(c_{e,i}, P) = g(c_i, P)$ when P is in e_i, $= 0$ otherwise.) To be precise, $\tilde{g}$ depends on the range of c; when not specified, c will range over B.

A subset H of B^* is called total on B if for every $v \in B$ with $v \neq 0$ there is some $c \in H$ with $cv \neq 0$. For such H a function $g(P)$, valued in B, is called *BH-integrable* if $g(P)c$ is measurable and integrable on every e for every c in the closed linear subspace $\bar{H}$ determined by H; then G is called the *BH-integral* of g if $G(e)$ is valued in B and for every c in $\bar{H}$ and every e,

$$G(e)c = \int_e g(P)c\, d\gamma(P).$$

$g(P)$, valued in B, is called *Bochner integrable* if it is Bochner measurable and $|g(P)|$ has finite integral on every e; then G, valued in B, is called the *Bochner integral* of g if for every e and every n there is a finitely valued $\sum_i c_{e,i}$ with disjoint $e_i \subset e$ such that

$$\gamma(e - \textstyle\sum_i e_i) < 1/n, \quad |g(P) - c_i| < 1/n$$

for all $P \in e_i$, and $|G(e) - \sum_i \gamma(e_i)c_i| < 1/n$.

We shall say that B *is separable-controlled by* H if H is a separable subset of B^* such that for every $v \in B$, $|v| = \sup|vc|$ for all $c \in H$ with $|c| = 1$ (cf. Pettis [**10**, p. 257]).

B is said to have *the RN property on* S if for every e of positive measure and every G valued in B and majorized on e, there exists, for every $\epsilon > 0$, a set e' of positive measure contained in e, and a c in B, such that

$$\left|\frac{G(e'')}{\gamma(e'')} - c\right| < \epsilon$$

for all e'' of positive measure contained in e'.

S is said to have a countable basis if there exists a sequence $e_1, e_2, \ldots$, such that for every e and every $\epsilon > 0$ there is a set e', the union of a subsequence of the e_n, such that $\gamma(e - ee') + \gamma(e' - ee') < \epsilon$.

3. Length functions. We refer to §2 of this paper and to [**7**, §2] for terminology.

It is easy to verify the following statements for an *arbitrary* length function λ: the conjugate λ^* is always continuous; $\lambda^{***} = \lambda^*$; if E is λ-null and $\gamma(E) > 0$ then E is λ^*-purely-infinite; if E is λ-purely-infinite then E is λ^*-null; $\lambda(u) = \lambda(u_1)$ whenever $u(P) = u_1(P)$ for almost all P; $\lambda(u) = 0$ if and only if $\{u(P) > 0\}$ is a λ-null set; $\lambda(u) < \infty$ implies $\{u(P) = \infty\}$ is a λ-null set; and $\lambda(u) \geqslant \lambda^{**}(u)$ for all u.

For an arbitrary length function λ the L^λ and, more generally, the $L^\lambda(B)$ are obviously linear, normed spaces with norm $\lambda(f)$, where f, f_1 are identified if and only if $\{f(P) \neq f_1(P)\}$ is a λ-null set. Completeness will be shown now by a variation of the von Neumann-Weyl argument [**13**, p. 111].

THEOREM 3.1. *$L^\lambda(B)$ is a Banach space.*

Proof. We need only show completeness. Given a sequence f_n with all $\lambda(f_n)$ finite and $\lambda(f_n - f_m) \to 0$ as $n, m \to \infty$, select an infinite subsequence g_i from the f_n such that

$$\sum_{i=1}^{\infty} \lambda(g_{i+1} - g_i)$$

is finite. For each P set

$$g_0(P) = |g_1(P)| + \sum_{i=1}^{\infty} |g_{i+1}(P) - g_i(P)|.$$

Then

$$\lambda(g_0) \leqslant \lambda(g_1) + \sum_{i=1}^{\infty} \lambda(g_{i+1} - g_i) < \infty$$

and hence $\{g_0(P) = \infty\}$ is a λ-null set E_0. Set $f(P) = 0$ for all P in E_0 and set

$$f(P) = g_1(P) + \sum_{i=1}^{\infty} (g_{i+1}(P) - g_i(P))$$

for all other P. Then $\lambda(f)$ is finite, $\lambda(f - g_i) \to 0$ as $i \to \infty$, and hence $\lambda(f - f_n) \leqslant \lambda(f - g_i) + \lambda(g_i - f_n) \to 0$ as n (and i) $\to \infty$.

Remark 1. Theorem 3.1 shows that the spaces $L_W{}^p(B)$, $L_{(w)}{}^p(B)$, and $M_{(w)}{}^q(B)$, in particular, are Banach spaces [cf. **7**, p. 276; **14**; **3a**, pp. 130, 131].

Remark 2. If f is in $L^\lambda(B)$ and E is λ-purely-infinite, then $f(P) = 0$ for almost all P in E.

4. Levelling length functions. We refer to §2 of this paper and to [**7**, §2] for terminology.

By repetition, (L 8) implies, under the same conditions, the more general

$$\text{(L 8)}' \qquad \lambda\Big(\sum_{i=1}^{m} k_{e_i}\Big) \geq \lambda\bigg(\bigg(\frac{k_1\gamma(e_1)+\ldots+k_r\gamma(e_r)}{\gamma(e_1+\ldots+e_r)}\bigg)_{e_1+\ldots+e_r} + \sum_{i=r+1}^{m} k_{e_i}\bigg)$$

for all $2 \leq r \leq m$. For a continuous length function, (L 8) actually implies, as can be verified without difficulty:

(L 8)″ *For every finite or countable collection of disjoint* e_i *with all* $\gamma(e_i) > 0$, $\lambda(u)$ *is not increased if on each* e_i, $u(P)$ *is replaced by its average on* e_i, *namely*

$$\frac{1}{\gamma(e_i)}\int_{e_i} u(P)\,d\gamma(P).$$

The particular case of (L 8) with $m = 2$ and $k_1 = 1$, $k_2 = 0$, together with (L 4) and (L 2) gives

$$\text{(L 8)}''' \qquad \lambda(e_1) \geq \frac{\gamma(e_1)}{\gamma(e_1+e_2)}\lambda(e_1+e_2) \geq \frac{\gamma(e_1)}{\gamma(e_1+e_2)}\lambda(e_2).$$

It is now easy to verify the following statements for an arbitrary levelling length function; $0 < \lambda(e) < \infty$ whenever $0 < \gamma(e) < \infty$; E is λ-null if and only if $\gamma(E) = 0$; $\lambda(u) = 0$ if and only if $u(P) = 0$ for almost all P; there are no λ-purely-infinite sets; if $\gamma < \infty$ then $\lambda(S) < \infty$ but if $\gamma = \infty$ then $0 < \lambda(E) = \lambda(S) \leq \infty$ for all E with $\gamma(E) = \infty$; $\lambda(u) < \infty$ implies that u has finite integral on every e; and if S is not coarse then $\gamma(e) > 0$, $\gamma(e) \to 0$ implies $\lambda(e) \to \lambda(+0) \geq 0$.

If λ is a levelling length function, then

$$\begin{aligned}
\lambda^*(e) &= \sup \int_e u(P)\,d\gamma(P), && \lambda(u) \leq 1,\\
&= \sup \int_e k\,d\gamma(P), && k \geq 0,\ k\lambda(e) \leq 1,\\
&= \gamma(e)\,\lambda(e)^{-1}, && \lambda(e) > 0.
\end{aligned}$$

Thus we have the identity:

$$\lambda(e)\,\lambda^*(e) = \gamma(e).$$

Throughout the remainder of this paper λ *will denote a levelling length function.*

THEOREM 4.1. (i) $\lambda^{**} = \lambda$.
(ii) λ^* *is a levelling length function.*
(iii) *the* $\lambda\lambda^*$*-Hölder inequality*

$$\text{(4.1)} \qquad \int u(P)v(P)\,d\gamma(P) \leq \lambda(u)\,\lambda^*(v)$$

holds, as well as the converses: the supremum of the left-hand side of (4.1) *for all* v *with* $\lambda^*(v) \leq 1$, *is* $\lambda(u)$; *and the supremum of the left-hand side of* (4.1) *for all* u *with* $\lambda(u) \leq 1$, *is* $\lambda^*(v)$. *Moreover, the first of these converses holds even if* v *is further restricted to be constant on each of any finite or countable collection of* e_i

on each of which the fixed u is constant and similarly the second converse holds even if u is further restricted to be constant on each of any finite or countable collection of e_i on which the fixed v is constant.

Proof of (i). For fixed m let $i = 1, \dots, m$ and let e_i be fixed, disjoint sets of positive measure. The finitely valued functions in L^λ of the form $\alpha = \sum_i \alpha_{e,i}$ form an m-dimensional Banach space H. The conjugate H^* can clearly be represented as the set of finitely valued functions $\beta = \sum_i \beta_{e,i}$ where $\beta\alpha$ shall mean $\sum_i \alpha_i \gamma(e_i) \beta_i$; the norm of β in H^* has the value sup $\sum_i \alpha_i \gamma(e_i) \beta_i$ for all $\lambda(\alpha) \leq 1$, and this equals $\lambda^*(\beta)$ since λ is a levelling length function and (L 8)$''$ is valid. Hence for every such α there is a β with $\lambda^*(\beta) = 1$ and $\beta\alpha = \lambda(\alpha)$; it follows easily that $\lambda^{**}(u) \geq \lambda(u)$ for all finitely valued u, and hence for all u. Since $\lambda(u) \geq \lambda^{**}(u)$ holds for every length function, this proves (i).

Proof of (ii) *and* (iii). Since $\lambda^*(e) = \gamma(e)/\lambda(e)$ and $0 < \lambda(e) < \infty$ whenever $0 < \gamma(e) < \infty$ it follows that λ^* satisfies (L 7). Using (L 8)$''$ for λ we can write

$$\tag{4.2} \lambda^*\left(\left(\frac{k_1 \gamma(e_1) + k_2 \gamma(e_2)}{\gamma(e_1 + e_2)}\right)_{e_1+e_2} + \sum_{i=3}^{m} k_{e,i}\right)$$
$$= \sup \left(\frac{k_1 \gamma(e_1) + k_2 \gamma(e_2)}{\gamma(e_1 + e_2)}\, \gamma(e_1 + e_2)\, h + \sum_{i=3}^{m} k_i \gamma(e_i)\, h_i\right)$$

for all non-negative h, h_i $(i \geq 3)$ with $\lambda(\sum_{i=1}^{m} h_{e,i}) \leq 1$ (where $h_1 = h_2 = h$). Since this supremum is not decreased when the condition $h_1 = h_2$ is removed, we find that

$$\text{the expression (4.2)} \leq \sup \sum_{i=1}^{m} k_i \gamma(e_i)\, h_i$$

for all non-negative h_i with

$$\lambda\left(\sum_{i=1}^{m} h_{e,i}\right) \leq 1.$$

Hence

$$\text{the expression (4.2)} \leq \lambda^*\left(\sum_{i=1}^{m} k_{e,i}\right),$$

showing that λ^* satisfies (L 8). This proves (ii), and (iii) follows easily.

We note that $|F|(e)$ is a numerically valued additive set function for which $|F|(e) = 0$ whenever $\gamma(e) = 0$, and hence $\lambda(|F|)$ is defined.

THEOREM 4.2. *For every F valued in B,*

(i) $$|F(e)| \leq |F|(e) \leq \lambda(F_e)\, \lambda^*(e) \leq \lambda(F)\, \lambda^*(e)$$

for every e,

(ii) $$\lambda(F) = \lambda(|F|).$$

Proof. Let m be fixed, $i = 1, \dots, m$ and suppose $e = e_1 + \dots + e_m$ with e_i disjoint sets of positive measure. Then

$$\lambda(F_e) \geqq \lambda \sum_i \left(\frac{|F(e_i)|}{\gamma(e_i)}\right)_{e_i} \geqq \lambda\left(\frac{\sum_i |F(e_i)|}{\gamma(e)}\right)_e$$
$$= \frac{\sum_i |F(e_i)|}{\gamma(e)} \lambda(e),$$

showing that $\sum_i |F(e_i)| \leqq \lambda(F_e)\, \lambda^*(e)$. This implies (i).

To prove (ii), since $\lambda(|F|) \geqq \lambda(F)$ is clear (in fact, $|F|(e) \geqq |F(e)|$ for every e), we need only prove $\lambda(|F|) \leqq \lambda(F)$. Hence we may suppose $\lambda(F)$, and by (i), all $|F|(e)$, to be finite. Now let m be fixed, $i = 1, \ldots, m$, and e_i disjoint sets of positive measure; then for every $\epsilon > 0$ there exist disjoint decompositions

$$e_i = \sum_j e_{ij} \qquad (j = 1, \ldots, m_i),$$

with $\gamma(e_{ij}) > 0$ for all i, j and

$$|F|(e_i) = \sum_j |F(e_{ij})| + \epsilon_i$$

with $0 \leqq \epsilon_i < \epsilon\gamma(e_i)$ for all i. Then

$$\lambda \sum_i \left(\frac{|F|(e_i)}{\gamma(e_i)}\right)_{e_i} \leqq \inf_{\epsilon>0} \left(\lambda \sum_i \left(\sum_j \frac{|F(e_{ij})|}{\gamma(e_i)}\right)_{e_i} + \epsilon \sum_i \lambda(e_i)\right)$$
$$\leqq \inf_{\epsilon>0} \left(\lambda \sum_i \sum_j \left(\frac{|F(e_{ij})|}{\gamma(e_{ij})}\right)_{e_{ij}} + \epsilon \sum_i \lambda(e_i)\right)$$
$$\leqq \lambda(F).$$

This implies $\lambda(|F|) \leqq \lambda(F)$ and completes the proof of (ii).

We note that for a levelling length function: each of (L 11) and (L 12) implies (L 10) but (L 10) need not imply either of (L 11) or (L 12); and (L 12) need not imply (L 11). One of (L 9), (L 9)* can fail to hold but at least one of them holds since $\lambda(e)\, \lambda^*(e) = \gamma(e)$.

THEOREM 4.3. *If* (L 9)* *holds and* $\lambda(F) < \infty$ *then*:

(i) $$F\left(\sum_{i=1}^{m} e_i\right) \to F(e) \quad \textit{and} \quad |F|\left(\sum_{i=1}^{m} e_i\right) \to |F|(e)$$

whenever $$e = \sum_{i=1}^{\infty} e_i.$$

(ii) *For every* $E = e_1 + e_2 + \ldots$ *there is a point function* $u(E) = u(E, P) \geqq 0$ *for all* P *for which* $\lambda(F_E) = \lambda(|F_E|) = \lambda(u(E))$ *and*

$$|F|(e) = \int_e u(E, P)\, d\gamma(P)$$

for all $e \subset E$.

(iii) $\lambda(F_{E(n)}) \to \lambda(F_E)$ *whenever*

$$E = \sum_{i=1}^{\infty} E_i$$

where $E(n)$ *denotes* $E_1 + \ldots + E_n$.

Proof. By Theorem 4.2 and the additivity of F and $|F|$, setting

$$e' = e - \sum_{i=1}^{m} e_i,$$

we obtain

$$|F(e) - F\left(\sum_{i=1}^{m} e_i\right)| = |F(e')| \leqslant \lambda(F)\,\lambda^*(e'),$$

and the same relation holds with $|F|$ in place of F. When $m \to \infty$, $\gamma(e') \to 0$ and (L 9)* implies that $\lambda^*(e') \to 0$. This proves (i).

The Radon-Nikodym theorem then shows that $u(E)$ exists with

$$|F|(e) = \int_e u(E, P)\, d\gamma(P)$$

for all $e \subset E$. By (L 8)'', $\lambda(F_E) \leqslant \lambda(u(E))$. To complete the proof of (ii) we need only show that for $e \subset E$ with $u(E, P)$ bounded on e, say $u(E, P) \leqslant K$ on e, $\lambda((u(E))_e) \leqslant \lambda(F_E)$. For this purpose, with fixed m and $i = 1, \dots, m$, let e_i be the subset of e on which $K(i-1)/m \leqslant u(E, P) \leqslant Ki/m$. Then for almost all P,

$$(u(E))_e(P) \leqslant \sum_{i=1}^{m}{}' \left(\frac{|F|(e_i)}{\gamma(e_i)}\right)_{e_i}(P) + \frac{K}{m},$$

where $\sum'$ indicates that the i for which $\gamma(e_i) = 0$ are to be omitted. Hence

$$\lambda((u(E))_e) \leqslant \lambda(F_E) + \frac{K}{m}\lambda(e)$$

for all m, implying $\lambda((u(E))_e) \leqslant \lambda(F_E)$ and proving (ii).

To prove (iii) we need only show that $\lambda(F_E) \leqslant \sup \lambda(F_{E(n)})$ for all n since $\lambda(F_{E(n)})$ is non-decreasing with increasing n and does not exceed $\lambda(F_E)$. We need therefore only show that for fixed m $(i = 1, \dots, m)$, and disjoint e_i of positive measure contained in E,

$$\lambda \sum_i \left(\frac{|F(e_i)|}{\gamma(e_i)}\right)_{e_i} \leqslant \sup \lambda(F_{E(n)})$$

for all n. Now by (i), for each i, $|F(e_{i,r})| \to |F(e_i)|$ as $r \to \infty$, where $e_{i,r}$ denotes $e_i(E_1 + \dots + E_r)$. Hence it is sufficient to show that for every r,

$$\lambda \sum_i \left(\frac{F(e_{i,r})}{\gamma(e_{i,r})}\right)_{e_{i,r}} \leqslant \lambda(F_{E(n)})$$

for some n; and this is clearly so for $n > r$.

Corollary 1. *If* (L 9)* *holds and* $\lambda(F_e)$ *is finite then* e *can be decomposed into a finite or countable number of disjoint* e_i *on each of which* F *is majorized.*

Corollary 2. *If* (L 9)* *holds and* $\lambda(F)$ *is finite and* (L 10)' *holds then there exists a decomposition* $S = E_0 + e_1 + e_2 + \dots$ *with disjoint* E_0, e_i *such that* F *is majorized on each* e_i *and vanishes on* E_0.

COROLLARY 3. *(L 11) and (L 9)* together imply (L 10)′ if λ is a levelling length function.*

5. $L^\lambda(B)^*$ in terms of set functions.

THEOREM 5.1. *If (L 12) and (L 13) hold, the relation*

$$G(e)c = \Phi(c_e) \tag{5.1}$$

for all c in B and all e, sets up a (1, 1) linear correspondence between all Φ in $L^\lambda(B)^*$ and all G, valued in B^*, for which $\lambda^*(G)$ is finite; for corresponding elements, $|\Phi| = \lambda^*(G)$.

Proof. For given Φ in $L^\lambda(B)^*$, $|\Phi(c_e)| \leq |\Phi||c| \lambda(e)$ and hence (5.1) defines $G(e)$, valued in B^*, as an additive set function with $G(e) = 0$ whenever $\gamma(e) = 0$. To see that $\lambda^*(G) \leq |\Phi|$, let $i = 1, \ldots, m$ for fixed m, and choose arbitrary disjoint e_i of positive measure, arbitrary c_i in B with $|c_i| = 1$ and arbitrary scalars α_i; set $f(P) = \sum_i \alpha_i c_{e,i}(P)$. Then f is in $L^\lambda(B)$, $\lambda(f) = \lambda(\sum_i \alpha_{e,i})$ and $|\Phi(f)| \leq |\Phi|\lambda(f)$; hence

$$\left|\sum_i \alpha_i\gamma(e_i)\beta_i\right| \leq |\Phi|\lambda\left(\sum_i \alpha_{e,i}\right), \quad \beta_i = \frac{G(e_i)c_i}{\gamma(e_i)};$$

the converse of the $\lambda\lambda^*$-Hölder inequality now shows that $\lambda^*(\sum_i \beta_{e,i}) \leq |\Phi|$. Since the c_i can be chosen to make $G(e_i)c_i$ arbitrarily near to $|G(e_i)|$ for each i,

$$\lambda^*\left(\sum_i \left(G(e_i)/\gamma(e_i)\right)_{e_i}\right) \leq |\Phi|$$

implying $\lambda^*(G) \leq |\Phi|$.

On the other hand, the $\lambda\lambda^*$-Hölder inequality shows that $|\Phi(f)| \leq \lambda^*(G)\,\lambda(f)$ for finitely valued f and (L 12), (L 13) imply that these f are dense in $L^\lambda(B)$. This means that $|\Phi| \leq \lambda^*(G)$ and so $|\Phi| = \lambda^*(G)$.

Conversely, for given G with $\lambda^*(G)$ finite, (5.1) defines $\Phi(c_e)$ for all c in B and all e; for finitely valued $f = \sum_i c_{e,i}$ we set $\Phi(f) = \sum_i \Phi(c_{e,i})$. Then $\Phi(f)$ is defined and

$$|\Phi(f)| \leq \lambda^*(G)\,\lambda(f)$$

for a set of f dense in $L^\lambda(B)$. It follows that $\Phi(f)$ extends uniquely to all f in $L^\lambda(B)$ so that the extension Φ is in $L^\lambda(B)^*$ and $|\Phi| \leq \lambda^*(G)$. The preceding two paragraphs now show that $|\Phi| = \lambda^*(G)$.

6. Set functions in terms of integrals.

We refer to §2 for terminology.

If $\mathfrak{g}$ is such that $g(c, P)$ is identical with $g(P)c$ for some point function $g(P)$ valued in B^*, for all c in B, and if $|g(P)|$ is measurable, then $\lambda^*(\mathfrak{g}) \leq \lambda^*(g)$ and inequality may actually occur. However, if the point function $g(P)$ is Bochner measurable, it is not difficult to verify that equality holds.

LEMMA 6.1. *Suppose G, valued in B^*, corresponds by the relation*

$$G(e)c = \int_e g(c, P)\, d\gamma(P) \tag{6.1}$$

for all c in B and all e, to a $\mathfrak{g} = g(c, P)$ for which

$$\int_e |g(c, P)|\, d\gamma(P)$$

is finite for every c and every e. Then

$$\lambda^*(G) = \lambda^*(\mathfrak{g}) < \infty. \tag{6.2}$$

Proof. When c varies over all elements in B with $|c| = 1$,

$$|G(e)| = \sup \left| \int_e g(c, P)\, d\gamma(P) \right| \leq \sup \int_e |g(c, P)|\, d\gamma(P).$$

(L 8)$''$ then implies that $\leq$ holds in (6.2). To prove that $\geq$ holds in (6.2), we need only show that for disjoint e_i and $|c_i| = 1$,

$$\lambda^* \sum_i g(c_{e,i}) \leq \lambda^*(G).$$

We need only consider the case that for every i, $k \leq |g(c_i, P)| \leq K$ for all P in e_i for some $0 < k < K < \infty$. Now suppose $0 < \eta < 1$ and decompose each e_i into disjoint subsets $e_i = \sum_j e_{ij}$ $(j = 1, \ldots, m_i)$, so that for each e_{ij}, $\gamma(e_{ij}) > 0$ and there is scalar h_{ij} such that for all P in e_{ij},

$$|g(c_i, P)| \geq |h_{ij}|, \quad |g(c_i, P) - h_{ij}| \leq \eta |h_{ij}|,$$

(this can clearly be done, using a partition of the complex numbers $k \leq |z| \leq K$ by a finite number of parallels to the axes of reals and pure imaginaries). Then for all P in e_{ij},

$$\begin{aligned} \frac{|G(e_{ij})|}{\gamma(e_{ij})} &\geq \frac{1}{\gamma(e_{ij})} \left(\int_{e_{ij}} |h_{ij}|\, d\gamma(P) - \int_{e_{ij}} |g(c_i, P) - h_{ij}|\, d\gamma(P) \right) \\ &\geq |h_{ij}|(1 - \eta) \geq (1 - \eta)(1 + \eta)^{-1} |g(c_i, P)|. \end{aligned}$$

This implies $\lambda^*(G) \geq (1 - \eta)(1 + \eta)^{-1}\ \lambda^* \sum_i g(c_{e,i})$ for all $0 < \eta < 1$ and hence $\geq$ holds in (6.2). This establishes (6.2).

Remark. If $|g(c, P)|$ has *finite* integral on every e for every c then there always exists a G, valued in B^* which corresponds to $\mathfrak{g}$ by (6.1). This is shown by the argument of [**5**, p. 308].

THEOREM 6.1. *If* (L 10)$'^*$ *holds or if S has property* (R) *then the relation* (6.1) *sets up a* (1, 1) *linear correspondence between the $\mathfrak{g}$ with $\lambda^*(\mathfrak{g})$ finite and the G, valued in B^* with $\lambda^*(G)$ finite; for corresponding elements $\lambda^*(G) = \lambda^*(\mathfrak{g})$.*

Proof. For given $\mathfrak{g}$, if $\lambda^*(\mathfrak{g})$ is finite, the $\lambda\lambda^*$-Hölder inequality shows that

$$\int_e |g(c, P)|\, d\gamma(P) \leq \lambda(e)\lambda^*(\mathfrak{g})|c|$$

for all c in B and all e; hence (6.1) defines G, valued in B^*, and Lemma 6.1 shows that $\lambda^*(G) = \lambda^*(\mathfrak{g})$.

Conversely, if G, valued in B^*, has $\lambda^*(G)$ finite, then

$$|G(e)c| < \lambda^*(G)\,\lambda(e)|c|$$

so that $G(e)c$ is, for fixed c, a numerically valued absolutely continuous set function. By the Radon-Nikodym theorem, for given $E = e_1 + e_2 + \ldots$ there exists an integrable function $g(E)(c, P)$ satisfying (6.1) for all $e \subset E$. If (L 10)$'^*$ holds or if S has property (R) then $\mathfrak{g} = g(c, P)$ may be defined by "combining" the $g(E)(c, P)$ so that $\mathfrak{g}$ will satisfy the requirements of the theorem.

THEOREM 6.2. *If* (L 10)$'^*$ *holds or if S has property* (R), *and if also* (L 12) *and* (L 13) *hold, then the relation*

$$\Phi(c_e) = \int_e g(c, P)\, d\gamma(P) \tag{6.3}$$

sets up a (1, 1) *linear correspondence between the Φ in $L^\lambda(B)^*$ and the $\mathfrak{g}$ with $\lambda^*(\mathfrak{g})$ finite; for corresponding elements* $|\Phi| = \lambda^*(\mathfrak{g})$.

Proof. This theorem is an immediate consequence of Theorems 6.1 and 5.1.

7. $L^\lambda(B)^*$ in terms of point functions. We refer to §2 for terminology; discussions of measurability, integrability and integral for vector-valued functions (and additional relevant references) are given in [**2**], [**5**], [**8**], and [**9**]. The BH-integral of $g(P)$, if it exists at all, will be unique since H is required to be total on B but the BH-integral may depend on B and H for a given g; however, it can be shown that every H^*H-integrable g (this is the special case of $B = H^*$) does have an H^*H-integral G [**5**, p. 308].

The function g is Bochner measurable if and only if g is locally almost-separable-valued, that is, for every e there is an e_0 of zero measure contained in e such that the set of values of $g(P)$, for all P in $e - e_0$, is a separable set and, for every c in this separable set, $|g(P) - c|$ is measurable; thus the Bochner measurability of g, though dependent on h is *not* dependent on B provided that for every e, almost all values of $g(P)$ for P in e are contained in B. If g is Bochner integrable, then the Bochner integral always exists, it is uniquely determined and it depends on g but not on B. Bochner integrability implies that the BH-integral exists, and coincides with the Bochner integral for every $H \subset B^*$ with H total on B.

If B is separable-controlled by H then B can be considered as part of $\bar{H}^*$ which will also be separable-controlled by H; thus B is separable-controlled if and only if B can be imbedded isometrically in the conjugate to a separable Banach space. Every separable B is, of course, separable-controlled. If B is separable-controlled by H then B is separable-controlled by a suitable countable subset of elements of norm 1 contained in H.

LEMMA 7.1. *Suppose* (L 9) *and either* (L 10)$'^*$ *or property* (R) *for S hold. If $\lambda^*(G)$ is finite and $V(G)$ is separable-controlled by a Banach space H then G is the H^*H-integral of some g, valued in H^*, such that $|g(P) - h|$ is measurable for*

every h in H^* and $\lambda^*(g) = \lambda^*(\mathfrak{g}) = \lambda^*(G)$ where $\mathfrak{g} = g(c) = g(c, P) = g(P)c$ and c varies over H.

LEMMA 7.2. *Suppose* (L 9) *and either* (L 10)'* *or property* (*R*) *for* S *hold. If $\lambda^*(G)$ is finite and $V(G) \subset B^*$ for some separable B^*, then there is a $g(P)$, valued in B^*, such that G is the Bochner integral of g and $\lambda^*(g) = \lambda^*(G)$.*

LEMMA 7.3. *Suppose* (L 9) *and either* (L 10)'* *or property* (*R*) *for* S *hold. If $\lambda^*(G)$ is finite and, for every e, $V(G_e)$ is separable and locally weakly compact, then there is a $g(P)$, valued in $V(G)$ such that G is the Bochner integral of g and $\lambda^*(g) = \lambda^*(G)$.*

Remark. If (L 9) holds and $\lambda^*(G)$ is finite and e has a countable basis, then $V(G_e)$ is necessarily separable.

Proof of Lemma 7.1. Suppose G majorized by M on some e. Since G may be considered as valued in H^*, Theorem 6.1 implies the existence of a $\mathfrak{g}$ such that *for every c* in H,

$$\left| \int_{e_1} g(c, P)\, d\gamma(P) \right| = |G(e_1)c| < M|c|\, \gamma(e_1)$$

for every $e_1 \subset e$. It follows that $|g(c, P)| < M|c|$ outside of a set of zero measure (depending on c) and we may suppose $g(c, P) = 0$ for P outside e.

Since H is separable there exists a countable set (c_k) of elements dense in H and including all finite rational linear combinations of its members. Then for every P outside a single set N of zero measure, $g(c_k, P)$ is rational linear in c_k and $|g(c_k, P)| < M|c_k|$ for all k. For fixed P set $g_1(c_k, P) = g(c_k, P)$ outside N, $= 0$ in N and for the remaining c in H, define $g_1(c, P)$ to be $\lim g_1(c^m, P)$ where c^m is any subsequence of the c_k with $c^m \to c$ as $m \to \infty$. It is easily verified that $g_1(c, P)$ is now uniquely defined, and that $|g_1(c, P)| < M|c|$ for all c and all P; therefore $g_1(c, P)$ can be expressed in the form $g(P)c$ with $g(P)$ valued in H^* and uniquely defined and with $|g(P)| < M$ for every P. Now for each c in H, $g(P)c = g_1(c, P)$ is measurable and

$$\begin{aligned}\int_e |g(P)c - g(c, P)|\, d\gamma(P) < {} & \int_e |g(P)(c - c_k)|\, d\gamma(P) \\ & + \int_e |g(P)c_k - g(c_k, P)|\, d\gamma(P) \\ & + \int_e |g(c - c_k, P)|\, d\gamma(P) \\ < {} & 2M\gamma(e)|c - c_k|\end{aligned}$$

for all k. This implies that for every fixed c in H, $g(P)c$ and $g(c, P)$ can be identified.

Corollaries 1 and 2 to Theorem 4.3 enable us to decompose S into sets e on which G is majorized, and we can do this in such a way, that by combining the $g(P)$ corresponding to these e, we get a single $g(P)$ which is H^*H-integrable

and has G as its H^*H-integral and for which $\lambda^*(\dot{g}) = \lambda^*(G)$ where $\dot{g}$ denotes $g(c, P) = g(P)c$.

Now let (c_j), with each $|c_j| = 1$, be a countable subset of H by which H^* is separable-controlled. Since $g(P)c$ is measurable for every c it follows that for every h in H^*, $(g(P) - h)c_j$ and $\sup |(g(P) - h)c_j|$ for all j, are measurable, and hence $|g(P) - h|$ is measurable. Furthermore $|g(P)| = \sup |g(c_j, P)|$ for all j, from which it follows that

$$\lambda^*(g) = \lambda^*(\dot{g}) = \lambda^*(G).$$

Proof of Lemma 7.2. Since B^* is separable-controlled by B, Lemma 7.1 applies (with B in place of H) and gives a $g(P)$ which is Bochner measurable since B^* is separable. Furthermore, $\lambda^*(g) < \infty$ implies that $|g(P)|$ has finite integral on each e and hence that g has a Bochner integral, necessarily coinciding with its B^*B-integral G. This proves the lemma.

Proof of Lemma 7.3. This follows from Lemma 7.2 since a separable and locally weakly compact space is reflexive [**1**, Théorème 13, p. 189] and hence $V(G_e)$ can be taken as B^* with $B = V(G_e)^*$.

THEOREM 7.1. *Suppose the following hold:* (L 9), *either* (L 10)'* *or property* (R) *for* S, (L 12), *and* (L 13). *If B is separable then the relation, for every f in* $L^\lambda(B)$,

$$\Phi(f) = \int f(P)g(P)\,d\gamma(P) \tag{7.1}$$

sets up a (1, 1) *linear correspondence between the Φ in $L^\lambda(B)^*$ and the B^*B-integrable $g(P)$ with finite $\lambda^*(g)$; for corresponding elements $\lambda^*(g) = |\Phi|$.*

Proof. To any Φ in $L^\lambda(B)^*$ corresponds a G, valued in B^* with $\lambda^*(G) = |\Phi|$ by Theorem 5.1. Since each $V(G)$ is separable-controlled by B, Lemma 7.1 gives the existence of a unique function $g(P)$ having G as B^*B-integral and with $\lambda^*(g) = \lambda^*(G) = |\Phi|$. Relation (7.1) follows at once for all finitely valued f. Since the finitely valued f are dense in $L^\lambda(B)$ and since, for such f,

$$\int |f(P)g(P)|\,d\gamma(P) \leq \lambda(f)\,\lambda^*(g),$$

it follows that the numerical valued function $f(P)g(P)$ is measurable and (7.1) holds for every f in $L^\lambda(B)$.

Conversely, suppose $g(P)$ is B^*B-integrable and $\lambda^*(g) < \infty$. Then for every finitely valued f, (7.1) defines $\Phi(f)$ so that $|\Phi(f)| \leq \lambda^*(g)\lambda(f)$ and Φ has a unique extension which satisfies (7.1) for all f in $L^\lambda(B)$.

THEOREM 7.2. *If B^* is separable and* (L 9), *either* (L 10)'* *or property* (R) *for* S, (L 12), *and* (L 13) *hold, then $L^\lambda(B)^* = L^\mu(B^*)$ where $\mu = \lambda^*$. If B^{**} is separable and* (L 9), (L 9)*, *either* (L 10)' *and* (L 10)'* *or property* (R) *for* S, (L 12), (L 12)*, (L 13), *and* (L 13)* *all hold then* $L^\lambda(B)^{**} = L^\lambda(B^{**})$.

Proof. For every G, valued in B^*, with $\lambda^*(G)$ finite, $V(G)$ is separable and separable-controlled by B [**1**, Théorème 12, p. 189]. Lemma 7.2 shows that $L^\lambda(B)^* \subset L^\mu(B^*)$. Since $L^\lambda(B)^* \supset L^\mu(B^*)$ is easily verified with the help of the $\lambda\lambda^*$-Hölder inequality, $L^\lambda(B)^* = L^\mu(B^*)$. The other parts of the theorem follow immediately from this.

THEOREM 7.3. *Suppose B is reflexive. If B is separable or if every e has a countable basis, and if* (L 9), *either* (L 10)$'^*$ *or property* (R) *for S,* (L 12), *and* (L 13) *hold, then $L^\lambda(B)^* = L^\mu(B^*)$; on the other hand, if* (L 9), (L 9)*, *either* (L 10)$'$ *and* (L 10)$'^*$ *or property* (R) *for S,* (L 12), (L 12)*, (L 13), *and* (L 13)* *all hold then $L^\lambda(B)^* = L^\mu(B^*)$ and $L^\lambda(B)$ is reflexive.*

Proof. The first part of the theorem follows from Theorem 5.1, Lemma 7.3, and the remark following Lemma 7.3, since for a Banach space, reflexitivity is equivalent to locally weak compactness [**6**].

To prove the second part of the theorem we need only show that if B is locally weakly compact a,d (L 9), (L 9)*, either (L 10)$'$ and (L 10)$'^*$ or property (R) for S, (L 12), (L 12)*, (L 13), (L 13)* all hold, then $L^\lambda(B)$ is locally weakly compact; indeed it is sufficient to show, under these conditions, that any sequence of finitely valued functions $f_n = \sum_i c_{n,e,i}$ with all $\lambda(f_n) < 1$ has a subsequence which converges weakly. But these f_n may be considered as valued in B_1 where B_1 is a suitable separable subspace of B. Since the first part of this theorem shows that $L^\lambda B_1)$ is reflexive, hence locally weakly compact, the theorem follows.

Theorems 7.1, 7.2, and 7.3 were developed in the classical $L^p(B)$ case, by Bochner and Taylor [**3**], Pettis [**10**], Dunford [**5**], Phillips [**11**; **12**], and Dieudonné [**4**].

8. The RN property.

We refer to §2 for terminology.

THEOREM 8.1. *For given B and S let λ vary over all levelling length functions satisfying* (L 9), (L 12), (L 13), *and, if S does not have the property* (R), (L 10)$'^*$. *Then the relation $L^\lambda(B)^* = L^\mu(B^*)$, with $\mu = \lambda^*$, holds for one of these λ if and only if it holds for all of them and if and only if B^* has the RN property on S. If B is reflexive then B has the RN property on every S.*

Proof of necessity. If G, valued in B^*, is majorized on e then G_e is related by (5.1) to a Φ in $L^\lambda(B)^*$, hence to a Bochner measurable $g(P)$ which has G_e as its Bochner integral. Since g can be approximated by finitely valued functions the RN property can be established.

Proof of sufficiency. Any Φ in $L^\lambda(B)^*$ corresponds by (5.1) to some G, valued in B^*, with $\lambda^*(G)$ finite. It is sufficient to show that if G is majorized by M on some e, then there is a g such that G_e is the Bochner integral of g. Now by the RN property, for fixed n, e can be decomposed into a finite or countable number of disjoint e_{ni} of positive measure such that, for some c_{ni},

$$\left|\frac{G(e')}{\gamma(e')} - c_{ni}\right| < \frac{1}{n}$$

for all $e' \subset e_{ni}$ with $\gamma(e') > 0$. Define $g_n(P)$ as the countably valued function with value c_{ni} on e_{ni} for all n and i. Let N be union of all intersections of any finite collection of the e_{ni} which have zero measure. Then N is also a set of zero measure, and for any P not in N and any n, m there is a set of positive measure $e' = e_{ni}\, e_{mj}$ containing P; hence

$$|g_n(P) - g_m(P)| < \left|\frac{G(e')}{\gamma(e')} - g_n(P)\right| + \left|\frac{G(e')}{\gamma(e')} - g_m(P)\right| < \frac{1}{n} + \frac{1}{m}$$

so that $g_n(P)$ converges uniformly, for all P outside N, to a limit $g(P)$. Clearly g is Bochner measurable and Bochner integrable.

Now for any e' and any n, let

$$e_{nk}' = e' \sum_{i=1}^{k} e_{ni}.$$

Then for every n,

$$\begin{aligned} \left|G(e') - \int_{e'} g(P)\, d\gamma(P)\right| &< \sup_k \left|G(e_{nk}') - \int_{e_{nk}'} g_n(P)\, d\gamma(P)\right| \\ &\qquad + \int_{e'} |g_n(P) - g(P)|\, d\gamma(P) \\ &< \frac{2}{n}\gamma(e). \end{aligned}$$

This shows that G_e is the Bochner integral of g as required and proves the sufficiency part of the theorem.

9. The $L_{(w)}{}^p(B)$ and $M_{(w)}{}^q(B)$ spaces. We refer to [**7**, §2] and to §2 of this paper for terminology.

Theorem 5.1 of [**7**] states that $\lambda_{(w)}{}^{p*} = \mu_{(w)}{}^q$ and it is not difficult to verify, with the help of the remark following Theorem 5.1 of [**7**], that $\lambda_{(w)}{}^p$ satisfies (L 8). It follows that $\lambda_{(w)}{}^p$ and $\mu_{(w)}{}^q$ are conjugate levelling length functions. They have the following properties:

(L 9) holds for $\lambda_{(w)}{}^p$ except when S fails to be coarse with $p = \infty$ (the proof is easy).

(L 9) holds for $\mu_{(w)}{}^q$ except when S fails to be coarse with $q = \infty$ and with $w^*(x)$ bounded (from the relation $\mu(e) = \gamma(e)/\lambda(e)$).

(L 10) holds for $\lambda_{(w)}{}^p$ for all p in Case (C_1) and for $1 < p < \infty$ in Case (C_2) [**7**, Corollary to Theorem 5.6]. (L 10)′ holds in these cases also.

(L 10) holds for $\mu_{(w)}{}^q$ except when $q = \infty$ in Case (C_2) [**7**, Corollary to Theorem 5.7]. (L 10)′ holds in these cases also.

(L 11) holds for $\lambda_{(w)}^p$ only in Case (C_1) with $1 < p < \infty$ and $w^*(x) > 0$ for all $0 < x < \gamma$ and in Case (C_2) with $1 < p < \infty$ [7, Theorem 5.3(i)].

(L 11) holds for $\mu_{(w)}^q$ except when $q = \infty$ [7, Theorem 5.3(ii)].

(L 12) holds for $\lambda_{(w)}^p$ only in Case (C_1) with $1 < p < \infty$ and in Case (C_2) with $1 < p < \infty$ [7, Theorem 5.6].

(L 12) holds for $\mu_{(w)}^q$ except when $q = \infty$ in Case (C_2) [7, Theorem 5.7].

(L 13) holds for $\lambda_{(w)}^p$ in all cases [7, Theorem 5.4(i)].

(L 13) holds for $\mu_{(w)}^q$ except when $q = \infty$ with $w^*(x)$ unbounded [7, Theorem 5.5(i)].

By specializing Theorems 7.1, 7.2, and 7.3 to these particular spaces we obtain:

(i) Suppose B is separable and specialize Theorem 7.1. Then: (α) $L_{(w)}^p(B)^*$ can be identified with the Banach space of $g(P)$ valued in B^* which are B^*B-integrable and have finite norm $\mu_{(w)}^q(g)$ in the following cases: Case (C_1) with $1 < p < \infty$, Case (C_1) with $p = \infty$ and S coarse, Case (C_2) with $1 < p < \infty$ (assuming, if $p = 1$, that S has property (R)); (β) $M_{(w)}^q(B)^*$ can be identified with the Banach space of $f(P)$ valued in B^* which are B^*B-integrable and have finite norm equal to $\lambda_{(w)}^p(f)$ in the following cases: Case (C_1) with either $1 < q < \infty$ or $q = \infty$ and S coarse, Case (C_2) with $1 < q < \infty$ (assuming, if $q = 1$, that S has property (R)), Case (C_3) with $1 < q < \infty$ (assuming for all $1 < q < \infty$, that S has property (R)), and Case (C_3) with $q = \infty$ and S coarse (assuming that S has property (R)).

(ii) Suppose B^* is separable and specialize Theorem 7.2. Then $L_{(w)}^p(B)^* = M_{(w)}^q(B^*)$ and $M_{(w)}^q(B)^* = L_{(w)}^p(B^*)$ in the cases detailed in (α), (β) respectively of the (i) preceding.

(iii) Suppose B^{**} is separable and specialize Theorem 7.2. Then $L_{(w)}^p(B)^{**} = L_{(w)}^p(B^{**})$ and $M_{(w)}^q(B)^{**} = M_{(w)}^q(B^{**})$ in the cases: Case (C_1) with $1 < p < \infty$, Case (C_2) with $1 < p < \infty$, and Case (C_1) with S coarse and either $p = 1$ or $p = \infty$.

(iv) Suppose either every e has a countable basis or B is separable. If B is reflexive then, specializing Theorem 7.3, $L_{(w)}^p(B)^* = M_{(w)}^q(B^*)$ in the cases listed in (α) of (i) preceding and $M_{(w)}^q(B)^* = L_{(w)}^p(B^*)$ in the cases listed in (β) of (i) preceding.

(v) Specialize Theorem 7.3. If B is reflexive then $L_{(w)}^p(B)$ and $M_{(w)}^q(B)$ are reflexive conjugate spaces in the cases listed in (iii) preceding.

A more detailed study of Cases (C_1) and (C_2) with $p = 1$ or $p = \infty$ and of Case (C_3) with $1 < p < \infty$ will be given in a subsequent publication.

Added Sept. 15, 1953. The discussion given above actually applies to *arbitrary* length functions if the following changes are made:

(i) If G is an additive set function, valued in B^*, defined for all e with $\lambda(e)$ finite and such that $G(e) = 0$ whenever $\lambda(e) = 0$, let $\lambda^*(G)$ be defined as

sup $\sum_i |G(e_i)|\alpha_i$ for all finite collections of disjoint e_i and non-negative α_i for which $\lambda(\sum_i \alpha_{e_i})< 1$.

(ii) (L 9) shall read: either S is coarse or for every E with $\lambda(E) < \infty$, $\lambda(e) \to 0$ whenever $\gamma(e) \to 0$ with $e < E$.

(iii) (L 12) shall read: $\lambda(u) < \infty$, $\epsilon > 0$ imply $\lambda(u - u_e) < \epsilon$ for some e with $\lambda(e) < \infty$.

Then $\lambda^*(G) = \lambda^*(|G|)$ and $|G|(e) < \lambda^*(G)\lambda(e)$. If $\lambda^*(G) < \infty$ then G is absolutely continuous on every e with $\lambda(e) < \infty$; Corollaries 1 and 2 to Theorem 4.3 continue to hold if a λ-purely-infinite set E_∞ may be added to the decompositions.

Now Theorems 5.1, 6.1, 6.2, Lemmas 7.1, 7.2, 7.3 and Theorems 7.1, 7.2, 7.3 and 8.1 continue to hold for arbitrary length functions (not required to be levelling). Note that for Theorems 7.2 and 7.3 we use the additional assumption:

(L 14) Every non-negative measurable function $u(P)$ can be expressed as $u = u_1 + u_2$ with $\lambda^{**}(u) = \lambda^{**}(u_1) = \lambda(u_1)$ and $\lambda^{**}(u_2) = 0$.

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Queen's University

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AUTHOR'S MANUAL

UNIVERSITY OF TORONTO PRESS

TORONTO, 1953

AUTHOR'S MANUAL

THE University of Toronto Press has recently installed a new method of setting mathematical formulae with a view to reducing the hand work involved and improving the appearance of the finished product. The time and cost of setting are both materially reduced. First and second order indices, being set in place mechanically, are aligned with an accuracy almost impossible to achieve with hand work.

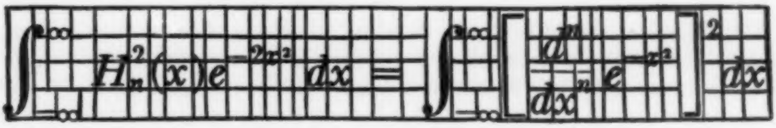
With the advent of *4-line typesetting* of formulae the careful preparation of a manuscript for submission to the printer has become of even greater importance than formerly. The task of the Editors will be greatly simplified and a cleaner manuscript for submission to the printer will result from a fuller understanding by the author of the difficulties involved. To this end authors are requested to co-operate by typing papers designed for publication in the *Canadian Journal of Mathematics* according to the simple rules laid down herein.

1. 4-LINE TYPESETTING. In the older method of Monotype composition of formulae each character was cast on a body 12 points high, and while this allowed for the setting of indices by machine, it precluded the superposition of superscripts and subscripts except by expensive handwork. To overcome this and other difficulties the new system allows for the casting of characters on 6-pt. bodies. With this added flexibility it is possible to build up even the most complicated formulae on the machine and avoid almost all the hand work which formerly led to delays and inaccuracies. The accompanying illustration shows the positions of the characters on the type bodies. All of the type is put in place mechanically except the large signs, brackets, and braces.

For the benefit of those unfamiliar with Monotype setting we briefly describe the mechanical processes involved. The edited manuscript is handed to the operator of a key-board much like that of a large typewriter. When a given key is pressed a certain combination of holes is punched in a paper tape. After the whole manuscript is keyboarded the tape is taken to a second machine called the *caster*. In this machine the code punched on the paper tape is translated into a pair of coordinates which locate a matrix or mould of the required symbol which is brought into position over a nozzle through which comes molten metal under pressure, thus casting the symbol. The metal hardens very rapidly and the symbol on its metal "body" is moved over into a tray which gradually builds up type to the length of a "galley." After the necessary changes are made by hand the metal is inked and a proof can be pulled.

It should be noted in the illustration below that characters frequently "overhang" and are supported on neighbouring bodies. That such support is adequate and no breakage of type occurs is a tribute to the accuracy with which the machinery is made. The need for such supporting bodies, usually blanks, places a considerable added strain on the keyboard operator who must visualize the arrangement of the symbols as they will issue from the caster.

$$\int_{-\infty}^{\infty} H_n^2(x)e^{-2x^2}\,dx = \int_{-\infty}^{\infty}\left[\frac{d^n}{dx^n}e^{-x^2}\right]^2 dx$$



2. Typing the Manuscript. The inclusion of indices of the first and second orders and other special symbols in the matrix case can only be made at some sacrifice of other characters. In consequence it is necessary for the Editors:

(*a*) *To divide the manuscript in two parts,* the first part to be cast from the "text" case using 12-pt. bodies and the second from the formula or "display" case using 6-pt. bodies. Text and formulae are, therefore, set on different keyboards.

(*b*) *To list the symbols used in a given paper,* so that as many as possible can be inserted in one or other or possibly both matrix cases, and hand work thus be avoided.

The whole Roman alphabet is available *only* in the text case, although italics are, of course, available in both. Thus it is possible to draw up the following suggestions for the typing of the manuscript.

Rule 1: If superscripts and subscripts are to be superposed, or if second order indices or fractions (not using a solidus) appear, then this material should be set as "display" (centred as formulae) and the manuscript typed accordingly.

Rule 2: Ordinary text (set in Roman type) should *not* be combined with displayed material as defined above.

3. General Rules. In addition to the important distinction between text and displayed material the following general rules for the preparation of manuscripts should be carefully followed.

(*a*) All manuscripts should be typed double-spaced on opaque paper, leaving good margins on both sides. Never type on both sides of a sheet of paper. The ***original*** and not a carbon copy should be submitted for publication.

(*b*) Sectional headings should be inserted but not underlined.

(*c*) Sufficient space should be left for displayed formulae, special symbols, etc., which are to be inserted by hand, so that they can be made sufficiently large to be clear to the printer. Particular attention should be paid to consistency as regards the formation of symbols; any doubt about the author's meaning may lead to confusion.

(*d*) Diagrams should be drawn on white bristol board or on linen in India ink, with a view to a reduction of 2 to 1 or 3 to 1 in the engraving. It should be kept in mind that this reduction will apply also to the thickness of the lines and to the lettering. A common fault is to make the lettering too small; another is to draw lines so close to one another that they run together when reduced. Both drawings and letters should be kept ***open***.

(*e*) To indicate italics, underline with a straight line; to indicate bold type, underline with a wavy line. Take care that such underlining does not run on underneath symbols or numerals, whether typed or handwritten. Do ***not*** underline isolated letters, x, y, P, Q, etc. which appear in the text or in displayed formulae, since these are always printed in italics. If other type is desired it should be indicated (see below).

(*f*) Underline Greek letters in red; type German letters and underline with green (or some colour other than red). If small capitals are desired underline with two straight lines, large capitals with three straight lines. It will save the author delay and the Editors much trouble if only those types and symbols are used which are available at the University of Toronto Press (see list below). If other symbols are desired it may be necessary to charge the cost of purchase to the author.

4. References and Footnotes. It is the practice of the ***Canadian Journal of Mathematics*** to list references in alphabetical order at the end of the paper. The references are then numbered and referred to by means of these numbers enclosed in square brackets. In order to reduce the number of footnotes, references should be inserted in the text. If a page reference is required it can also be enclosed within the square brackets; for example

[**2**, p. 25; **3**, p. 25, Theorem 5].

Footnotes should be indicated by superscripts, and numbered from the beginning of the paper. They should be typed on a separate sheet and not inserted at the foot of a page. Care should be taken to place the reference superscript on a word or at the end of a sentence, *not* on a symbol or formula.

In the following sample references it will be observed that titles, whether of papers or books, are printed in italics.

1. R. Brauer, *A note on Hilbert's Nullstellensatz*, Bull. Amer. Math. Soc., *54* (1948), 894-896.
2. P. Dubreil, *Algèbre*, vol. 1 (Paris, 1946).

5. SYMBOLS. Certain specific limitations of the ordinary typewriter may make mathematical copy difficult for the typesetter to interpret.

(*a*) If there is any possibility of doubt be careful to distinguish zero from the letters "o" and "O" by underlining the letters in pencil. A similar confusion is likely to arise between the Arabic one and the letter "ell." A typewriter may have the symbol 1 for one; if not, be sure to distinguish "one" from "ell" by superimposing a solidus to denote "ell."

(*b*) The letters "l" and "e," "r" and "n," "k" and "κ," "v" and "ν" must be carefully distinguished when handwritten. Capitals and small letters are also easily confused. Care should be taken to make clear whether a superscript ^{1} or a prime $'$ is meant, and whether a subscript $_1$ or a comma , .

(*c*) It is impossible to lay down hard-and-fast rules concerning the writing of mathematical formulae. From Rule 1, §2, however, it is clear that a solidus instead of a horizontal bar should be used in writing indices or fractions which are to be included in the text. The use of a negative index frequently leads to real simplification of a formula. The following examples will serve to indicate accepted usage and may suggest other simplifications in particular cases.

$$e^{(s-s_0)/c}, \quad K(r) = [1 + e^{-r}]^{-1}, \quad \frac{a+b}{2} = \tfrac{1}{2}(a+b),$$

$$f(x) > 2^{-n} f_n(x), \ \frac{dt}{ds} = c\left(1 - \frac{2m}{r}\right)^{-1}, \ a = \tfrac{3}{4}R_p + \tfrac{1}{3}R_q,$$

$$\prod_{j=1}^{p^r-1} \binom{(k-1)p^r + jp - 1}{p_* - 1} \Big/ \binom{jp-1}{p-1}, \quad \sum_{i=1}^{2k} (2^{-j} \nu n^{-1} - x)^i$$

TYPE FACES AND SYMBOLS

The University of Toronto Press has available the complete Latin alphabet, in capitals and lower case, in italic (A,a), Roman (A,a), and boldface (**A**,**a**) in 10-pt. and 8-pt. type. The Latin alphabet is also available in 6-pt. lightface italics and roman if required.

Papers for publication in the *Canadian Journal of Mathematics* are set in 10-pt. type, footnotes in 8-pt. type, tables in 10-pt. or 8-pt.

Special Characters Available in 10 pt. Latin Alphabet

1. Lightface italic, barred ($\bar{A}$,$\bar{a}$). Entire alphabet.
2. Lightface italic, with tilde ($\tilde{A}$,$\tilde{a}$). Entire alphabet.
3. Lightface italic, dotted. $\dot{a}\ \dot{e}\ \dot{g}\ \dot{m}\ \dot{n}\ \dot{q}\ \dot{r}\ \dot{s}\ \dot{u}\ \dot{x}\ \dot{y}\ \dot{z}$
4. Lightface italic, double dotted. $\ddot{a}\ \ddot{e}\ \ddot{g}\ \ddot{\imath}\ \ddot{o}\ \ddot{r}\ \ddot{u}\ \ddot{x}\ \ddot{y}\ \ddot{z}$
5. Lightface italic, underscored. $\underline{l}\ \underline{b}\ \underline{d}\ \underline{s}\ \underline{t}$
6. Script. $\mathcal{A}\mathcal{B}\mathcal{C}\mathcal{D}\mathcal{E}\mathcal{F}\mathcal{G}\mathcal{H}\mathcal{I}\mathcal{J}\mathcal{K}\mathcal{L}\mathcal{M}\mathcal{N}\mathcal{O}\mathcal{P}\mathcal{Q}\mathcal{R}\mathcal{S}\mathcal{T}\mathcal{U}\mathcal{V}\mathcal{W}\mathcal{X}\mathcal{Y}\mathcal{Z}$
7. Lightface italic indices (first order) $({}^{Aa}_{Aa})$. Entire alphabet.
8. Lightface italic indices (second order) $({}^{a^a})$. Entire alphabet lower case.
9. Overscored italic indices (first order) for use in denominator lines or as barred indices $({}^{\bar{a}})$. Entire alphabet lower case.
10. Lightface roman indices (first order). $^{\mathrm{exp}}\ ^{\mathrm{log}}$

Greek Alphabet

1. 10-pt. lightface

Alpha	α	A	Nu	ν	N
Beta	β	B	Xi	ξ	Ξ
Gamma	γ	Γ	Omicron	o	O
Delta	δ	Δ	Pi	π	Π
Epsilon	ϵ	E	Rho	ρ	P
Zeta	ζ	Z	Sigma	σ	Σ
Eta	η	H	Tau	τ	T
Theta	θ	Θ	Upsilon	υ	Υ
Iota	ι	I	Phi	ϕ	Φ
Kappa	κ	K	Chi	χ	X
Lambda	λ	Λ	Psi	ψ	Ψ
Mu	μ	M	Omega	ω	Ω

2. 8-pt. light face. Entire alphabet lower case.
3. 10-pt. lightface Greek overscored ($\bar{\alpha}$). All lower case. Also $\bar{\Delta}$ and $\bar{\Gamma}$.
4. 10-pt. lightface Greek with tilde ($\tilde{\alpha}$). All lower case.
5. 10-pt. lightface Greek dotted. $\dot{\alpha}\ \dot{\beta}\ \dot{\delta}\ \dot{\epsilon}\ \dot{\zeta}\ \dot{\eta}\ \dot{\psi}\ \dot{\xi}\ \dot{\theta}\ \dot{\vartheta}\ \dot{\gamma}\ \dot{\omega}$
6. 10-pt. lightface Greek double dotted. $\ddot{\alpha}\ \ddot{\epsilon}\ \ddot{\eta}\ \ddot{\iota}\ \ddot{o}\ \ddot{\upsilon}\ \ddot{\omega}$
7. Indices, first order $({}^{\alpha})$. All lower case.

Greek Characters (by number)

No. 3 Γ	No. 21 Φ	No. 31 η	No. 40 π	No. 49 α
" 4 Δ	" 22 X	" 32 θ	" 41 ρ	" 50 ϑ
" 8 Θ	" 23 Ψ	" 33 ι	" 42 σ	" 51 ϕ
" 11 Λ	" 24 Ω	" 34 κ	" 43 τ	" 52 ς
" 14 Ξ	" 26 β	" 35 λ	" 44 υ	" 54 $\varkappa$
" 16 Π	" 27 γ	" 36 μ	" 45 φ	" 55 ϱ
" 18 Σ	" 28 δ	" 37 ν	" 46 χ	" 56 ∂
" 19 T	" 29 ϵ	" 38 ξ	" 47 ψ	
" 20 ϒ	" 30 ζ	" 39 o	" 48 ω	

German Alphabet

1. 10-pt. lightface ($\mathfrak{a}$, $\mathfrak{A}$). Entire alphabet.

Numerals

1. Lightface indices, first order. ^{1234567890}
2. Lightface indices, first order, overscored. $^{\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}\bar{8}\bar{9}\bar{0}}$
3. Lightface indices, second order. $^{^{1234567890}}$
4. Fractional indices, first order. $^{\frac{1}{2}\frac{1}{3}}$
5. Fractional indices, first order, overscored. $^{\bar{\frac{1}{2}}}$
6. Piece fractions, in 10-pt. formulae. $\frac{1}{2}\frac{1}{3}\frac{1}{4}\frac{3}{4}\frac{4}{5}\frac{5}{6}\frac{6}{7}\frac{7}{8}\frac{8}{9}\frac{9}{10}$
7. 10-pt. numerals, overscored. $\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}\bar{8}\bar{9}\bar{0}$

Symbols

1. In 10-pt.

$= \quad \neq \quad \equiv \quad \not\equiv \quad \doteq \quad \approx \quad \doteqdot \quad \Bumpeq \quad \sim \quad \cong \quad \simeq \quad \nsim \quad \overline{\overline{\wedge}} \quad \overline{\wedge}$

$+ \quad \dot{+} \quad - \quad \pm \quad \underline{+} \quad \mp \quad \times \quad \div \quad \oplus \quad \ominus \quad \otimes \quad \odot$

$< \quad \nless \quad > \quad \ngtr \quad \lessdot \quad \gtrdot \quad \lesseqgtr \quad \gtreqless \quad \prec \quad \vee \quad \wedge \quad \ll \quad \gg$

$\subset \quad \supset \quad \subseteq \quad \supseteq \quad \not\subset \quad \not\supset \quad \cap \quad \cup \quad \in \quad \ni \quad \notin \quad \not\ni$

$\wp \quad \aleph \quad \hslash \quad \hbar \quad \infty \quad \square \quad \triangle \quad \angle \quad \perp \quad \propto \quad \oint \quad \therefore \quad ' \quad ^\circ \quad * \quad *$

$\rightarrow \quad \rightarrow \quad \leftarrow \quad \leftrightarrow \quad \rightleftarrows \quad \rightleftharpoons \quad /\!\!/ \quad \| \quad | \quad / \quad \checkmark \quad \surd \quad \{\,\} \quad (\,) \quad \langle\,\rangle \quad \{\,\} \quad [\,] \quad \Sigma \quad \int$

$$\int \Big\{ \Big\} \Big| \Big[\Big] \Big(\Big) \Sigma \quad \Pi \quad \cup \quad \cap \quad S$$

2. Indices, first order. $^{+ \, - \, - \, \infty \, \rightarrow \, > \, < \, > \, < \, = \, / \, , \, ' \, (\,) \, [\,] \, ' \, \cdot \, \aleph \, \surd}$

3. Indices, first order, overscored. $^{\bar{(} \, \bar{)} \, \bar{'} \, \mp \, = \, =}$

4. Indices, second order. $^{^{+ \, - \, -}}$

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